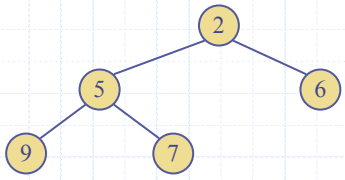
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

Heaps



Recall Priority Queue ADT

- A priority queue stores a collection of entries
- Each entry is a pair (key, value)
- Main methods of the Priority Queue ADT
  - insert(k, v)
    inserts an entry with key k and value v
  - removeMin()
    removes and returns the entry with smallest key
- Additional methods
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Recall PQ Sorting

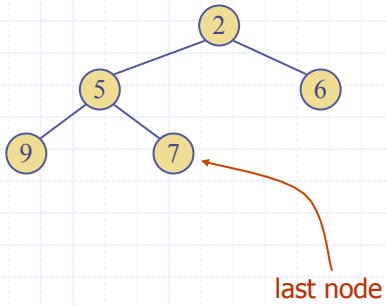
- We use a priority queue
  - Insert the elements with a series of insert operations
  - Remove the elements in sorted order with a series of removeMin operations
- The running time depends on the priority queue implementation:
  - Unsorted sequence gives selection-sort: $O(n^2)$ time
  - Sorted sequence gives insertion-sort: $O(n^2)$ time
- Can we do better?

Algorithm *PQ-Sort*(*S*, *C*)
Input sequence *S*, comparator *C* for the elements of *S*
Output sequence *S* sorted in increasing order according to *C*
P ← priority queue with comparator *C*
while ¬*S.isEmpty* ()
e ← *S.remove* (*S. first* ())
P.insert (*e*, *e*)
while ¬*P.isEmpty*()
e ← *P.removeMin*().*getKey*()
S.addLast(*e*)

Heaps

- A heap is a binary tree storing keys at its nodes and satisfying the following properties:
- Heap-Order: for every internal node v other than the root, $key(v) \geq key(parent(v))$
- Complete Binary Tree: let h be the height of the heap
  - for $i = 0, \ldots, h-1$, there are 2^i nodes of depth i
  - at depth $h-1$, the internal nodes are to the left of the external nodes

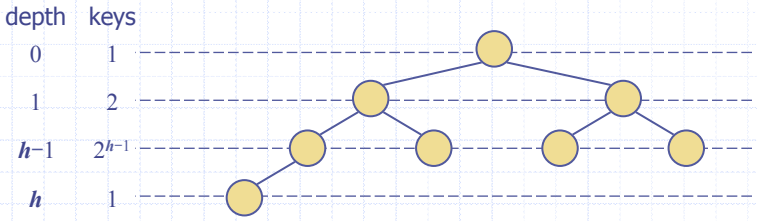
- The last node of a heap is the rightmost node of maximum depth



Height of a Heap

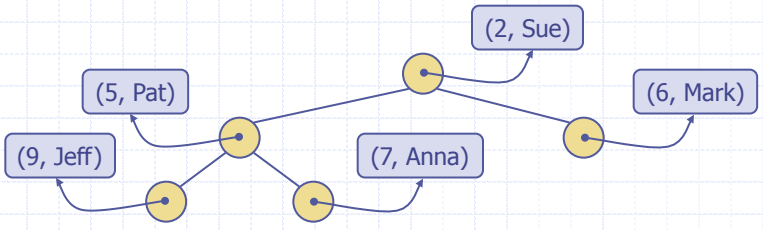
- Theorem: A heap storing n keys has height $O(\log n)$

  Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \ldots, h-1$ and at least one key at depth h, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-1} + 1$
  - Thus, $n \geq 2^h$, i.e., $h \leq \log n$



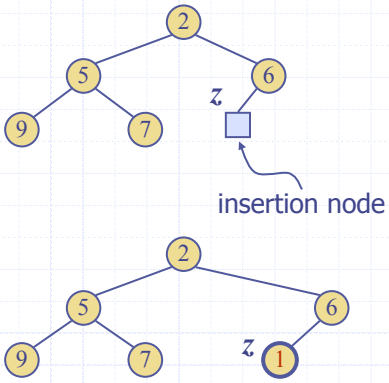
Heaps and Priority Queues

- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node



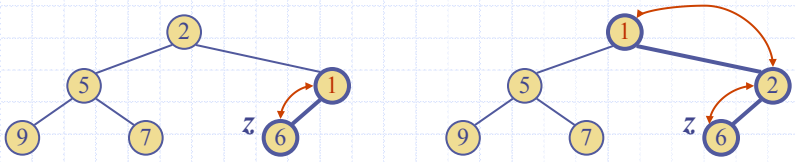
Insertion into a Heap

- Method insertItem of the priority queue ADT corresponds to the insertion of a key k to the heap
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
  - Store k at z
  - Restore the heap-order property (discussed next)



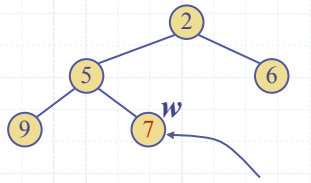
Upheap

- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time

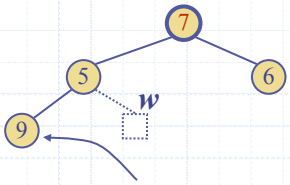


Removal from a Heap

- Method removeMin of the priority queue ADT corresponds to the removal of the root key from the heap
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node w
  - Remove w
  - Restore the heap-order property (discussed next)



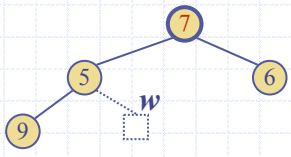
last node

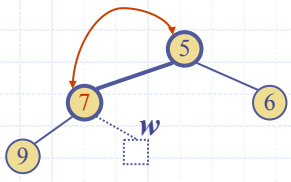


new last node

Downheap

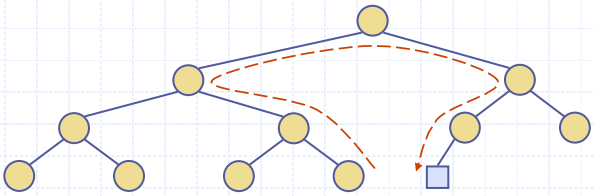
- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root
- Upheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time



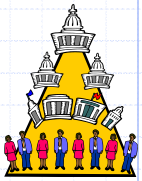


Updating the Last Node

- The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - Go up until a left child or the root is reached
  - If a left child is reached, go to the right child
  - Go down left until a leaf is reached
- Similar algorithm for updating the last node after a removal



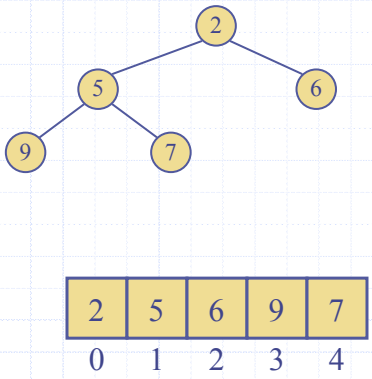
Heap-Sort



- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insert and removeMin take $O(\log n)$ time
  - methods size, isEmpty, and min take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

Array-based Heap Implementation

- We can represent a heap with n keys by means of an array of length n
- For the node at rank i
  - the left child is at rank $2i + 1$
  - the right child is at rank $2i + 2$
- Links between nodes are not explicitly stored
- Operation add corresponds to inserting at rank $n + 1$
- Operation remove_min corresponds to removing at rank n
- Yields in-place heap-sort



Java Implementation

```
1  /** An implementation of a priority queue using an array-based heap. */
2  public class HeapPriorityQueue<K,V> extends AbstractPriorityQueue<K,V> {
3    /** primary collection of priority queue entries */
4    protected ArrayList<Entry<K,V>> heap = new ArrayList<>();
5    /** Creates an empty priority queue based on the natural ordering of its keys. */
6    public HeapPriorityQueue() { super(); }
7    /** Creates an empty priority queue using the given comparator to order keys. */
8    public HeapPriorityQueue(Comparator<K> comp) { super(comp); }
9    // protected utilities
10   protected int parent(int j) { return (j-1) / 2; }          // truncating division
11   protected int left(int j) { return 2*j + 1; }
12   protected int right(int j) { return 2*j + 2; }
13   protected boolean hasLeft(int j) { return left(j) < heap.size(); }
14   protected boolean hasRight(int j) { return right(j) < heap.size(); }
15   /** Exchanges the entries at indices i and j of the array list. */
16   protected void swap(int i, int j) {
17     Entry<K,V> temp = heap.get(i);
18     heap.set(i, heap.get(j));
19     heap.set(j, temp);
20   }
21   /** Moves the entry at index j higher, if necessary, to restore the heap property. */
22   protected void upheap(int j) {
23     while (j > 0) {                    // continue until reaching root (or break statement)
24       int p = parent(j);
25       if (compare(heap.get(j), heap.get(p)) >= 0) break;   // heap property verified
26       swap(j, p);
27       j = p;                                  // continue from the parent's location
28     }
29   }
```

Java Implementation, 2

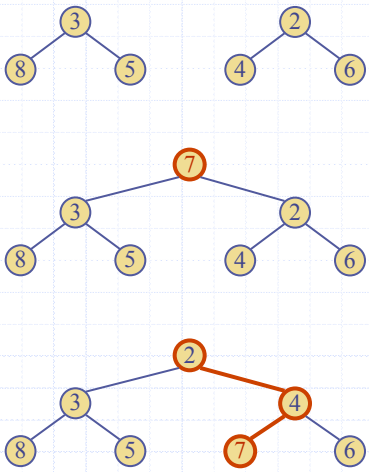
```
30   /** Moves the entry at index j lower, if necessary, to restore the heap property. */
31   protected void downheap(int j) {
32     while (hasLeft(j)) {                    // continue to bottom (or break statement)
33       int leftIndex = left(j);
34       int smallChildIndex = leftIndex;      // although right may be smaller
35       if (hasRight(j)) {
36         int rightIndex = right(j);
37         if (compare(heap.get(leftIndex), heap.get(rightIndex)) > 0)
38           smallChildIndex = rightIndex;     // right child is smaller
39       }
40       if (compare(heap.get(smallChildIndex), heap.get(j)) >= 0)
41         break;                              // heap property has been restored
42       swap(j, smallChildIndex);
43       j = smallChildIndex;                  // continue at position of the child
44     }
45   }
46
47   // public methods
48   /** Returns the number of items in the priority queue. */
49   public int size() { return heap.size(); }
50   /** Returns (but does not remove) an entry with minimal key (if any). */
51   public Entry<K,V> min() {
52     if (heap.isEmpty()) return null;
53     return heap.get(0);
54   }
```


Java Implementation, 3

```
55   /** Inserts a key-value pair and returns the entry created. */
56   public Entry<K,V> insert(K key, V value) throws IllegalArgumentException {
57     checkKey(key);        // auxiliary key-checking method (could throw exception)
58     Entry<K,V> newest = new PQEntry<>(key, value);
59     heap.add(newest);                          // add to the end of the list
60     upheap(heap.size( ) − 1);                  // upheap newly added entry
61     return newest;
62   }
63   /** Removes and returns an entry with minimal key (if any). */
64   public Entry<K,V> removeMin( ) {
65     if (heap.isEmpty( )) return null;
66     Entry<K,V> answer = heap.get(0);
67     swap(0, heap.size( ) − 1);                 // put minimum item at the end
68     heap.remove(heap.size( ) − 1);             // and remove it from the list;
69     downheap(0);                               // then fix new root
70     return answer;
71   }
72 }
```

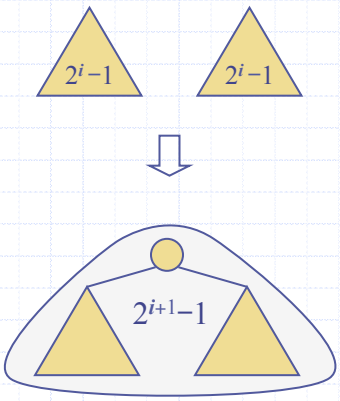
Merging Two Heaps

- We are given two two heaps and a key k
- We create a new heap with the root node storing k and with the two heaps as subtrees
- We perform downheap to restore the heap-order property

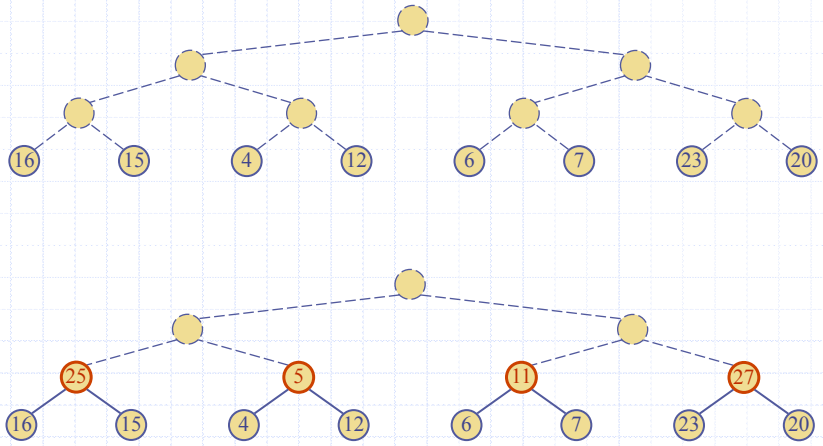


Bottom-up Heap Construction

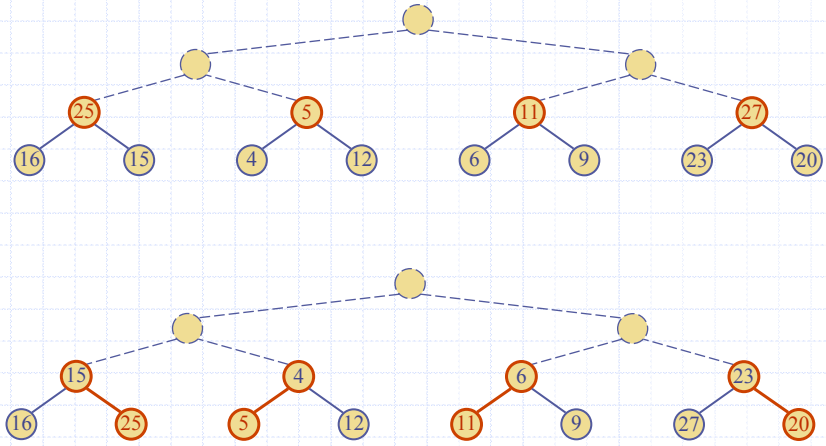
- We can construct a heap storing n given keys in using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1} - 1$ keys



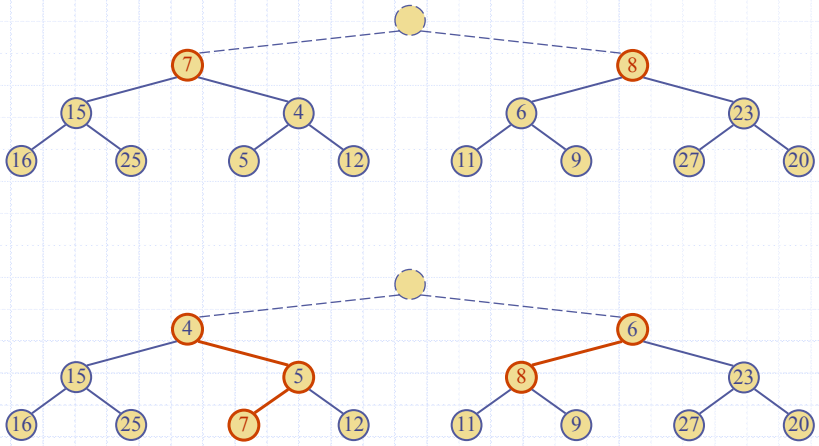
Example



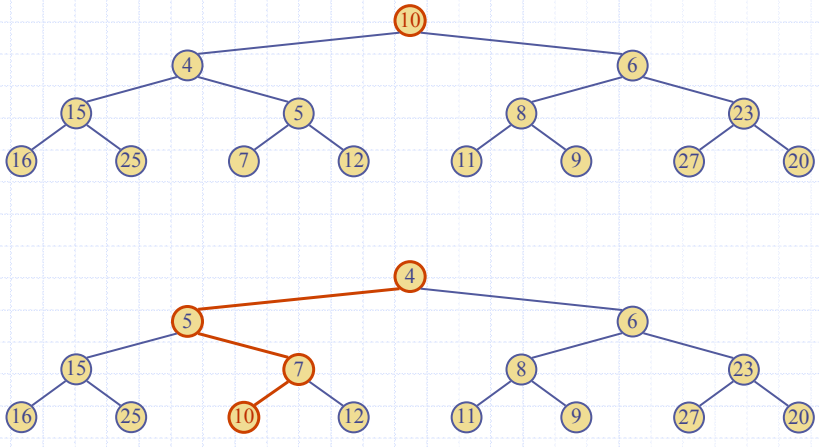
Example (contd.)



Example (contd.)

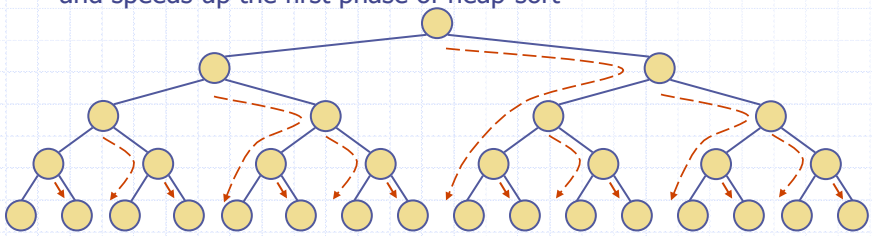


Example (end)



Analysis

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort



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