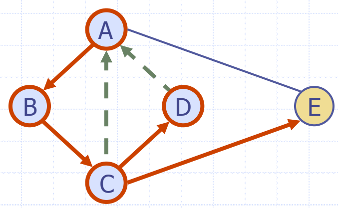
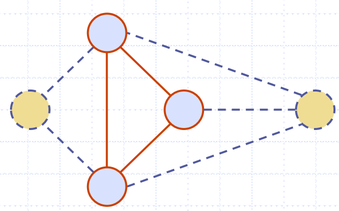
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

Depth-First Search

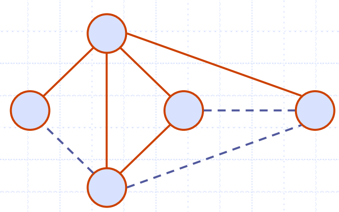


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



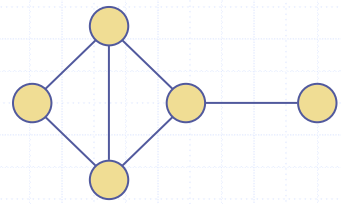
Subgraph



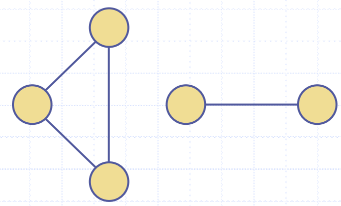
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



Connected graph



Non connected graph with two connected components

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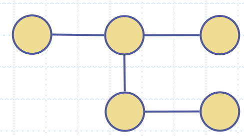
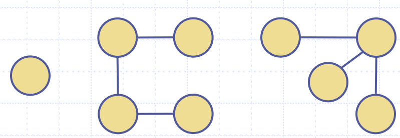
Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles

  This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



Tree



Forest

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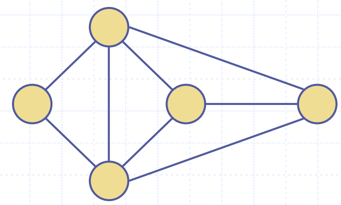
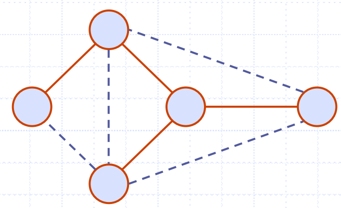
Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs what Euler tour is to binary trees

DFS Algorithm from a Vertex

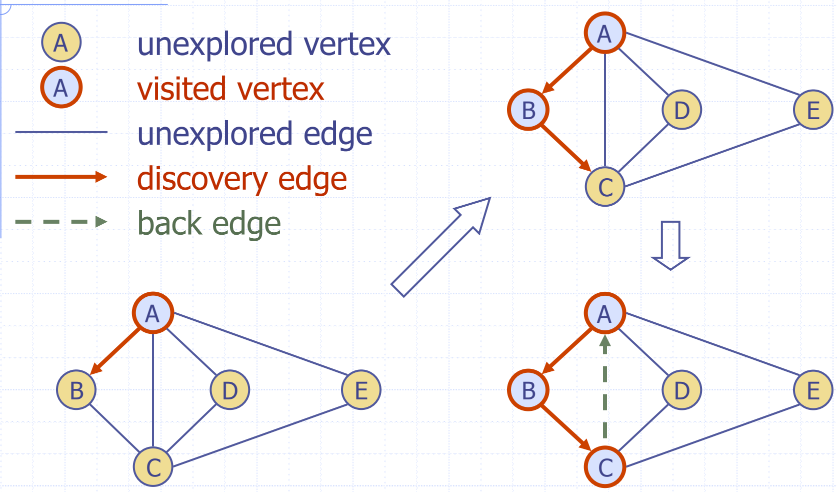
Algorithm DFS(G, u):
Input: A graph G and a vertex u of G
Output: A collection of vertices reachable from u, with their discovery edges

```
Mark vertex u as visited.
for each of u's outgoing edges, e = (u,v) do
   if vertex v has not been visited then
      Record edge e as the discovery edge for vertex v.
      Recursively call DFS(G, v).
```

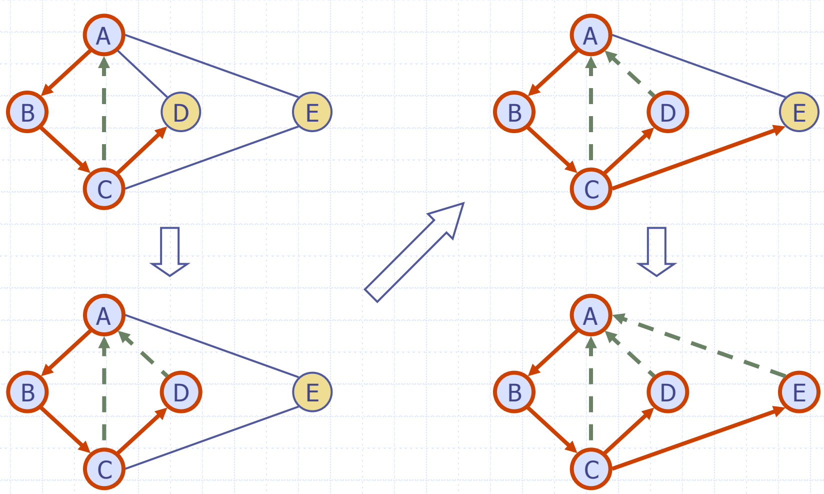

Java Implementation

```java
/** Performs depth-first search of Graph g starting at Vertex u. */
public static <V,E> void DFS(Graph<V,E> g, Vertex<V> u,
                 Set<Vertex<V>> known, Map<Vertex<V>,Edge<E>> forest) {
  known.add(u);                              // u has been discovered
  for (Edge<E> e : g.outgoingEdges(u)) {     // for every outgoing edge from u
    Vertex<V> v = g.opposite(u, e);
    if (!known.contains(v)) {
      forest.put(v, e);                      // e is the tree edge that discovered v
      DFS(g, v, known, forest);              // recursively explore from v
    }
  }
}
```

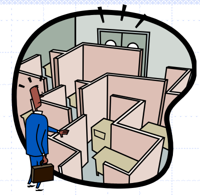

Example



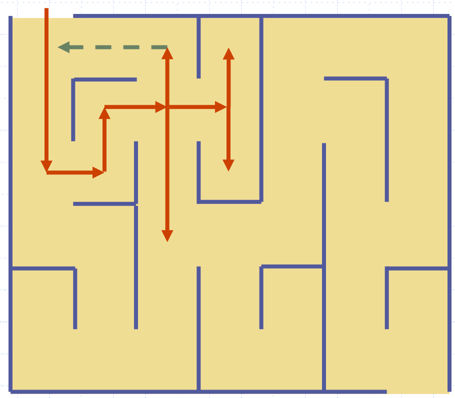
Example (cont.)



DFS and Maze Traversal



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



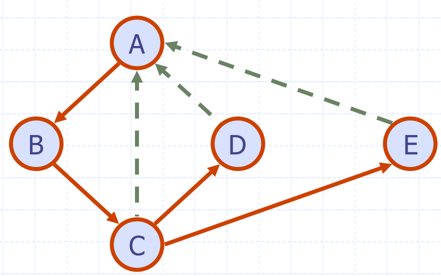
Properties of DFS

Property 1

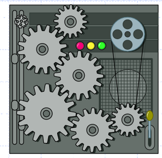
$DFS(G, v)$ visits all the vertices and edges in the connected component of v

Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of v



Analysis of DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $DFS(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop(e)
      else
        setLabel(e, BACK)
  S.pop(v)
```

Path Finding in Java

```
/** Returns an ordered list of edges comprising the directed path from u to v. */
public static <V,E> PositionalList<Edge<E>>
constructPath(Graph<V,E> g, Vertex<V> u, Vertex<V> v,
                    Map<Vertex<V>,Edge<E>> forest) {
  PositionalList<Edge<E>> path = new LinkedPositionalList<>( );
  if (forest.get(v) != null) {                    // v was discovered during the search
    Vertex<V> walk = v;                           // we construct the path from back to front
    while (walk != u) {
      Edge<E> edge = forest.get(walk);
      path.addFirst(edge);                        // add edge to *front* of path
      walk = g.opposite(walk, edge);              // repeat with opposite endpoint
    }
  }
  return path;
}
```


Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  for all  e ∈ G.incidentEdges(v)
    if  getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      S.push(e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        pathDFS(G, w, z)
        S.pop(e)
      else
        T ← new empty stack
        repeat
          o ← S.pop()
          T.push(o)
        until o = w
        return T.elements()
  S.pop(v)
```


DFS for an Entire Graph

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm DFS(G)
    Input graph G
    Output labeling of the edges of G
        as discovery edges and
        back edges
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
        DFS(G, v)
```

```
Algorithm DFS(G, v)
  Input graph G and a start vertex v of G
  Output labeling of the edges of G
    in the connected component of v
    as discovery edges and back edges
  setLabel(v, VISITED)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        DFS(G, w)
      else
        setLabel(e, BACK)
```


All Connected Components

- Loop over all vertices, doing a DFS from each unvisted one.

```java
/** Performs DFS for the entire graph and returns the DFS forest as a map. */
public static <V,E> Map<Vertex<V>,Edge<E>> DFSComplete(Graph<V,E> g) {
  Set<Vertex<V>> known = new HashSet<>();
  Map<Vertex<V>,Edge<E>> forest = new ProbeHashMap<>();
  for (Vertex<V> u : g.vertices())
    if (!known.contains(u))
      DFS(g, u, known, forest);          // (re)start the DFS process at u
  return forest;
}
```