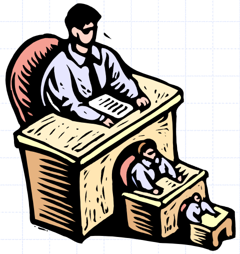
Using Recursion



Recall the Recursion Pattern (§ 2.5)

- **Recursion**: when a method calls itself
- Classic example - the factorial function:
  - n! = 1· 2· 3· ··· · (n-1)· n
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

- As a Java method:

```
// recursive factorial function
public static int recursiveFactorial(int n) {
   if (n == 0) return 1;      // basis case
   else return n * recursiveFactorial(n- 1); // recursive case
}
```

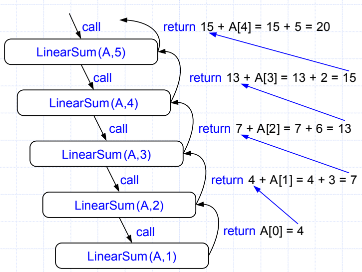

Linear Recursion (§ 4.1.1)

- **Test for base cases.**
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls **must** eventually reach a base case, and the handling of each base case should not use recursion.
- ***Recur once.***
  - Perform a single recursive call. (This recursive step may involve a test that decides which of several possible recursive calls to make, but it should ultimately choose to make just one of these calls each time we perform this step.)
  - Define each possible recursive call so that it makes progress towards a base case.

A Simple Example of Linear Recursion

Algorithm LinearSum(*A*, *n*):
Input:
A integer array *A* and an integer *n* = 1, such that *A* has at least *n* elements
Output:
The sum of the first *n* integers in *A*
if *n* = 1 **then**
return *A*[0]
else
return LinearSum(*A*, *n* - 1) + *A*[*n* - 1]

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(*A, i, j*):

Input: An array *A* and nonnegative integer indices *i* and *j*

Output: The reversal of the elements in *A* starting at index *i* and ending at *j*

```
if i < j then
    Swap A[i] and A[ j]
    ReverseArray(A, i + 1, j - 1)
return
```


Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional paramaters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(*A, i, j*), not ReverseArray(*A*).

Computing Powers

- The power function, p(x,n)=x^n, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x, n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$

$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$

$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$

$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$

A Recursive Squaring Method

Algorithm Power(x, n):
  Input: A number x and integer $n = 0$
  Output: The value x^n
  if $n = 0$ **then**
    return 1
  if n is odd **then**
    $y = \text{Power}(x, (n - 1)/2)$
    return $x \cdot y \cdot y$
  else
    $y = \text{Power}(x, n/2)$
    return $y \cdot y$

Analyzing the Recursive Squaring Method

Algorithm Power(x, n):
  Input: A number x and integer $n = 0$
  Output: The value x^n
  if $n = 0$ **then**
    return 1
  if n is odd **then**
    $y = \text{Power}(x, (n - 1)/2)$
    return $x \cdot y \cdot y$
  else
    $y = \text{Power}(x, n/2)$
    return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we used a variable twice here rather than calling the method twice.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(A, i, j):
  Input: An array A and nonnegative integer indices i and j
  Output: The reversal of the elements in A starting at index i and ending at j
  while $i < j$ **do**
    Swap $A[i]$ and $A[j]$
    $i = i + 1$
    $j = j - 1$
  return

Binary Recursion (§ 4.1.2)

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example: the DrawTicks method for drawing ticks on an English ruler.

```
---- 0
-
--
-
---
-
--
-
---- 1
-
--
-
---
-
--
-
---- 2
```

```
----- 0
-
--
-
---
-
--
-
----
-
--
-
---
-
--
-
----- 1
```

```
--- 0
-
--
-
--- 1
-
--
-
--- 2
-
--
-
--- 3
```

A Binary Recursive Method for Drawing Ticks

```
    // draw a tick with no label
public static void drawOneTick(int tickLength) { drawOneTick(tickLength, - 1); }
    // draw one tick
public static void drawOneTick(int tickLength, int tickLabel) {
  for (int i = 0; i < tickLength; i++)
     System.out.print("-");
  if (tickLabel >= 0) System.out.print(" " + tickLabel);
  System.out.print("\n");
}
public static void drawTicks(int tickLength) { // draw ticks of given length
  if (tickLength > 0) {              // stop when length drops to 0
     drawTicks(tickLength- 1);       // recursively draw left ticks
     drawOneTick(tickLength);        // draw center tick
     drawTicks(tickLength- 1);       // recursively draw right ticks
  }
}
public static void drawRuler(int nInches, int majorLength) { // draw ruler
  drawOneTick(majorLength, 0);       // draw tick 0 and its label
  for (int i = 1; i <= nInches; i++)  {
     drawTicks(majorLength- 1);      // draw ticks for this inch
     drawOneTick(majorLength, i);    // draw tick i and its label
  }
}
```

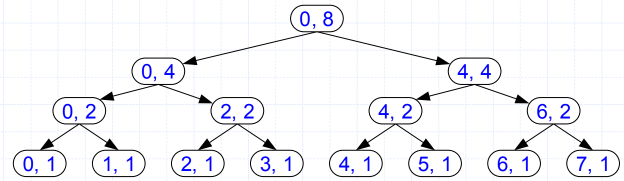
Note the two recursive calls

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

Algorithm BinarySum(A, i, n):
  Input: An array A and integers i and n
  Output: The sum of the n integers in A starting at index i
  if $n = 1$ **then**
    return $A[i]$
  return BinarySum(A, i, $n/2$) + BinarySum(A, $i + n/2$, $n/2$)

- **Example trace:**



Computing Fibanacci Numbers

- Fibonacci numbers are defined recursively:

$F_0 = 0$
$F_1 = 1$
$F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

- As a recursive algorithm (first attempt):

Algorithm BinaryFib(k):
  Input: Nonnegative integer k
  Output: The kth Fibonacci number F_k
  if $k = 1$ **then**
    return k
  else
    return BinaryFib(k - 1) + BinaryFib(k - 2)

Analyzing the Binary Recursion Fibonacci Algorithm

- Let n_k denote number of recursive calls made by BinaryFib(k). Then
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$.
- Note that the value at least doubles for every other value of n_k. That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead:

  Algorithm LinearFibonacci(k):
    Input: A nonnegative integer k
    Output: Pair of Fibonacci numbers (F_k, F_{k-1})
    if $k = 1$ **then**
      return (k, 0)
    else
      (i, j) = LinearFibonacci(k - 1)
      return ($i + j$, i)

- Runs in $O(k)$ time.

Multiple Recursion (§ 4.1.3)

- Motivating example: summation puzzles
  - *pot + pan = bib*
  - *dog + cat = pig*
  - *boy + girl = baby*

- Multiple recursion: makes potentially many recursive calls (not just one or two).

Algorithm for Multiple Recursion

Algorithm PuzzleSolve(k,S,U):
 Input: An integer k, sequence S, and set U (the universe of elements to test)
 Output: An enumeration of all k-length extensions to S using elements in U without repetitions
 for all e in U **do**
   Remove e from U {e is now being used}
   Add e to the end of S
   if k = 1 **then**
      Test whether S is a configuration that solves the puzzle
      if S solves the puzzle **then**
         return "Solution found: " S
   else
      PuzzleSolve(k - 1, S,U)
   Add e back to U {e is now unused}
   Remove e from the end of S

Visualizing PuzzleSolve