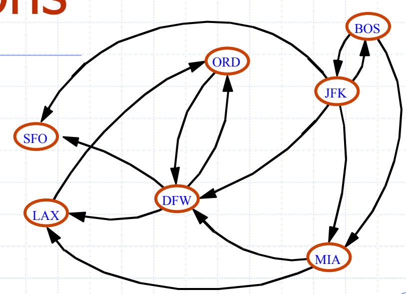
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

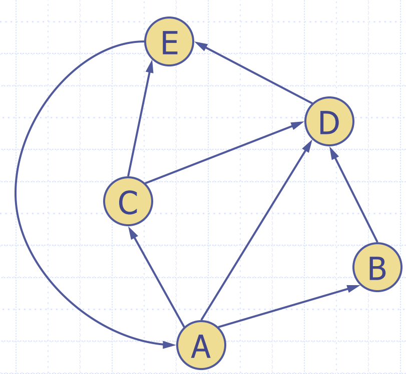
# Directed Graphs



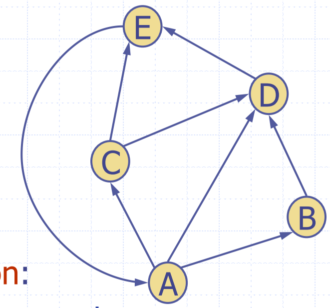
# Digraphs

- A digraph is a graph whose edges are all directed
  - Short for “directed graph”
- Applications
  - one-way streets
  - flights
  - task scheduling



# Digraph Properties

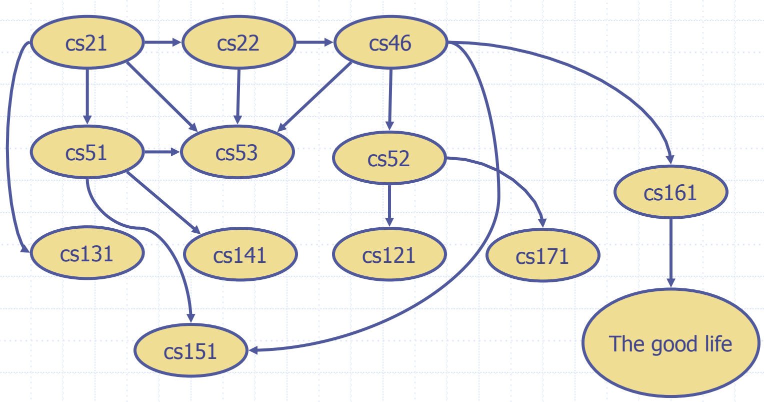



- A graph G=(V,E) such that
  - Each edge goes in one direction:
  - Edge (a,b) goes from a to b, but not b to a
- If G is simple, $m \leq n \cdot (n - 1)$
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of incoming edges and outgoing edges in time proportional to their size

# Digraph Application

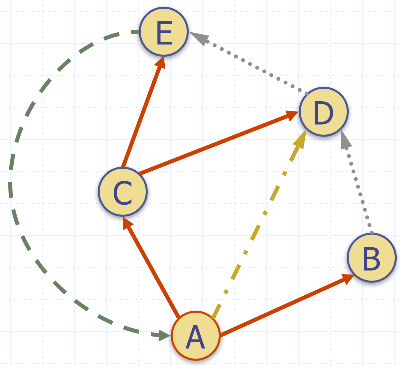
- Scheduling: edge (a,b) means task a must be completed before b can be started



# Directed DFS

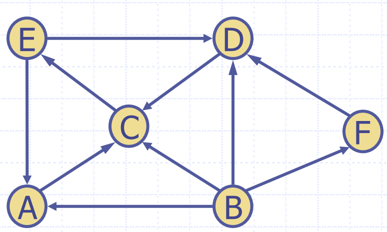
- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex $s$ determines the vertices reachable from $s$

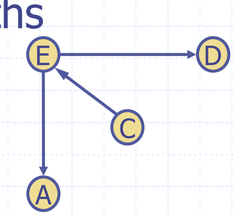


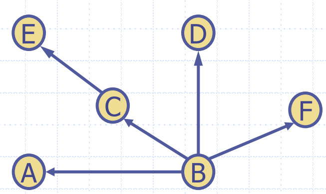



# Reachability

- DFS tree rooted at v: vertices reachable from v via directed paths

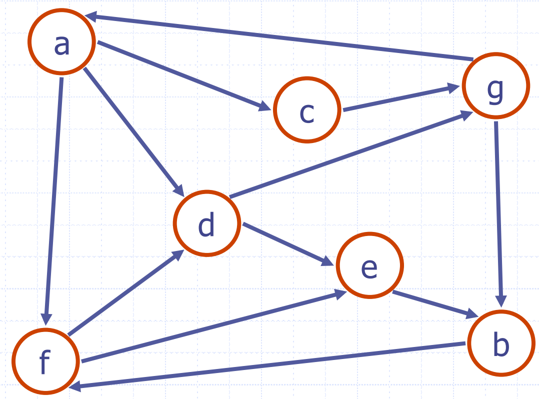


# Strong Connectivity

- Each vertex can reach all other vertices

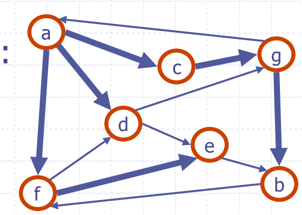




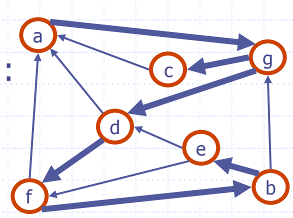

# Strong Connectivity Algorithm

- Pick a vertex v in G
- Perform a DFS from v in G
  - If there's a w not visited, print "no"
- Let G' be G with edges reversed
- Perform a DFS from v in G'
  - If there's a w not visited, print "no"
  - Else, print "yes"
- Running time: O(n+m)

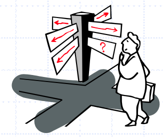
G:



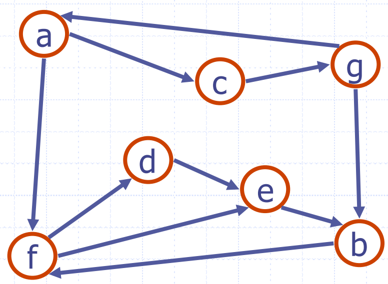

G' :



## Strongly Connected Components



- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in O(n+m) time using DFS, but is more complicated (similar to biconnectivity).



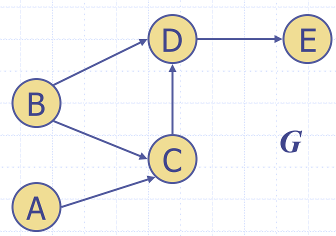

**{ a , c , g }**

**{ f , d , e , b }**

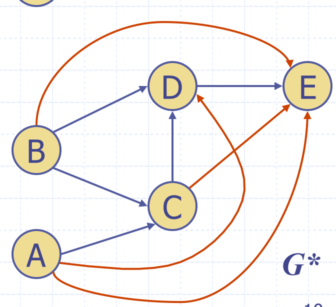


## Transitive Closure

- Given a digraph $G$, the transitive closure of $G$ is the digraph $G^*$ such that
  - $G^*$ has the same vertices as $G$
  - if $G$ has a directed path from $u$ to $v$ ($u \neq v$), $G^*$ has a directed edge from $u$ to $v$
- The transitive closure provides reachability information about a digraph




$G$

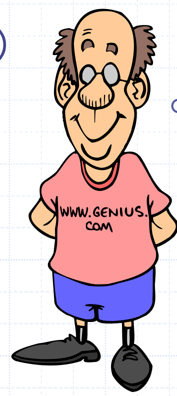



$G^*$

## Computing the Transitive Closure

- We can perform DFS starting at each vertex
  - O(n(n+m))

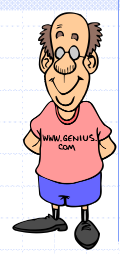
If there's a way to get from A to B and from B to C, then there's a way to get from A to C.




Alternatively ... Use dynamic programming: The Floyd-Warshall Algorithm

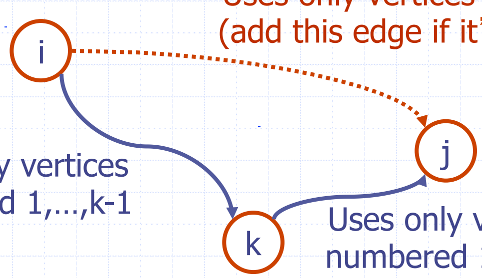


## Floyd-Warshall Transitive Closure



- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:

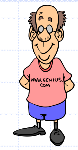
Uses only vertices numbered 1,…,k
(add this edge if it's not already in)




Uses only vertices numbered 1,…,k-1

Uses only vertices numbered 1,…,k-1

# Floyd-Warshall's Algorithm



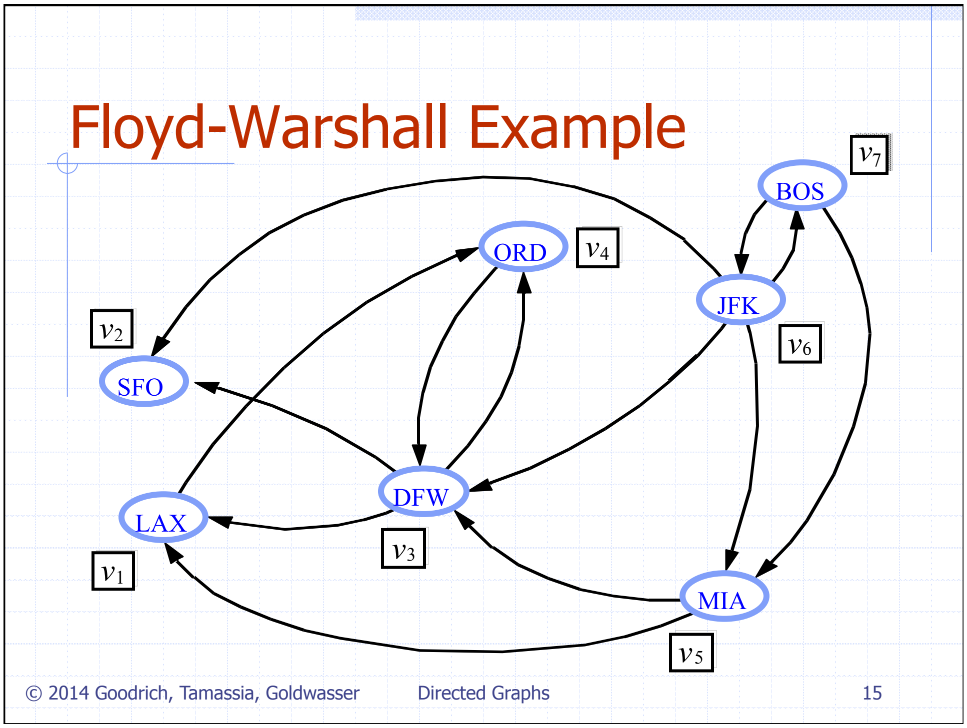
- Number vertices $v_1, \ldots, v_n$
- Compute digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - $G_k$ has directed edge $(v_i, v_j)$ if $G$ has a directed path from $v_i$ to $v_j$ with intermediate vertices in $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase $k$, digraph $G_k$ is computed from $G_{k-1}$
- Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

**Algorithm** *FloydWarshall*($G$)
  **Input** digraph $G$
  **Output** transitive closure $G^*$ of $G$
  $i \leftarrow 1$
  **for all** $v \in G.vertices()$
    denote $v$ as $v_i$
    $i \leftarrow i + 1$
  $G_0 \leftarrow G$
  **for** $k \leftarrow 1$ **to** $n$ **do**
    $G_k \leftarrow G_{k-1}$
    **for** $i \leftarrow 1$ **to** $n$ $(i \neq k)$ **do**
      **for** $j \leftarrow 1$ **to** $n$ $(j \neq i, k)$ **do**
        **if** $G_{k-1}.areAdjacent(v_i, v_k) \wedge$
            $G_{k-1}.areAdjacent(v_k, v_j)$
          **if** $\neg G_k.areAdjacent(v_i, v_j)$
            $G_k.insertDirectedEdge(v_i, v_j, k)$
    **return** $G_n$



# Java Implementation

```
/** Converts graph g into its transitive closure. */
public static <V,E> void transitiveClosure(Graph<V,E> g) {
  for (Vertex<V> k : g.vertices())
    for (Vertex<V> i : g.vertices())
      // verify that edge (i,k) exists in the partial closure
      if (i != k && g.getEdge(i,k) != null)
        for (Vertex<V> j : g.vertices())
          // verify that edge (k,j) exists in the partial closure
          if (i != j && j != k && g.getEdge(k,j) != null)
            // if (i,j) not yet included, add it to the closure
            if (g.getEdge(i,j) == null)
              g.insertEdge(i, j, null);
}
```


Floyd-Warshall Example
BOS
$v_7$
ORD
$v_4$
JFK
$v_6$
$v_2$
SFO
DFW
LAX
$v_3$
$v_1$
MIA
$v_5$
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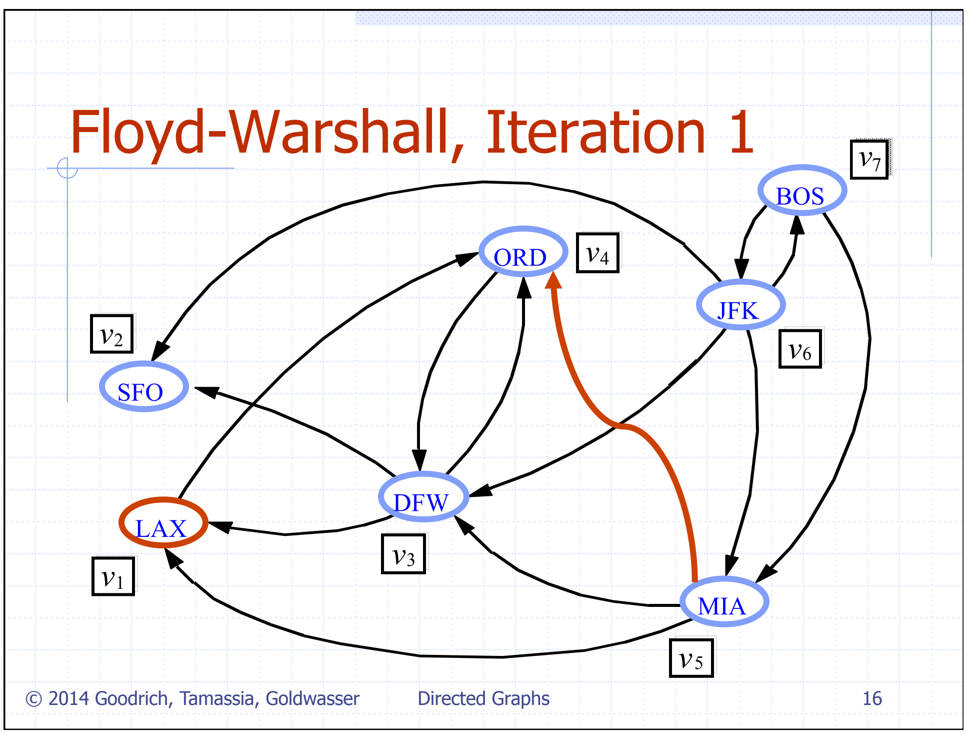
Floyd-Warshall, Iteration 1
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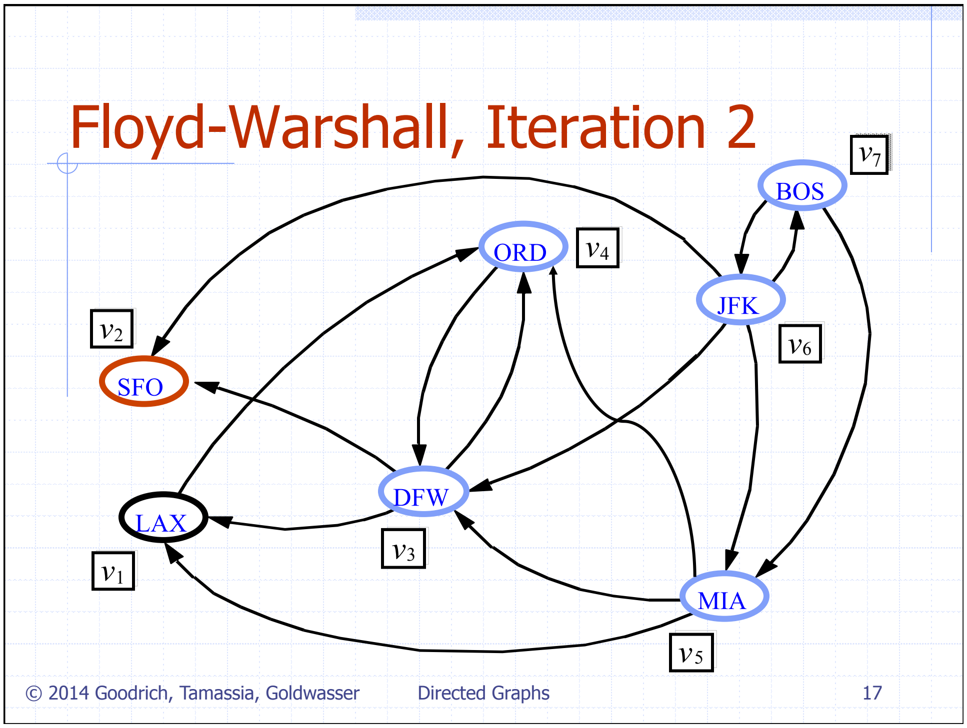
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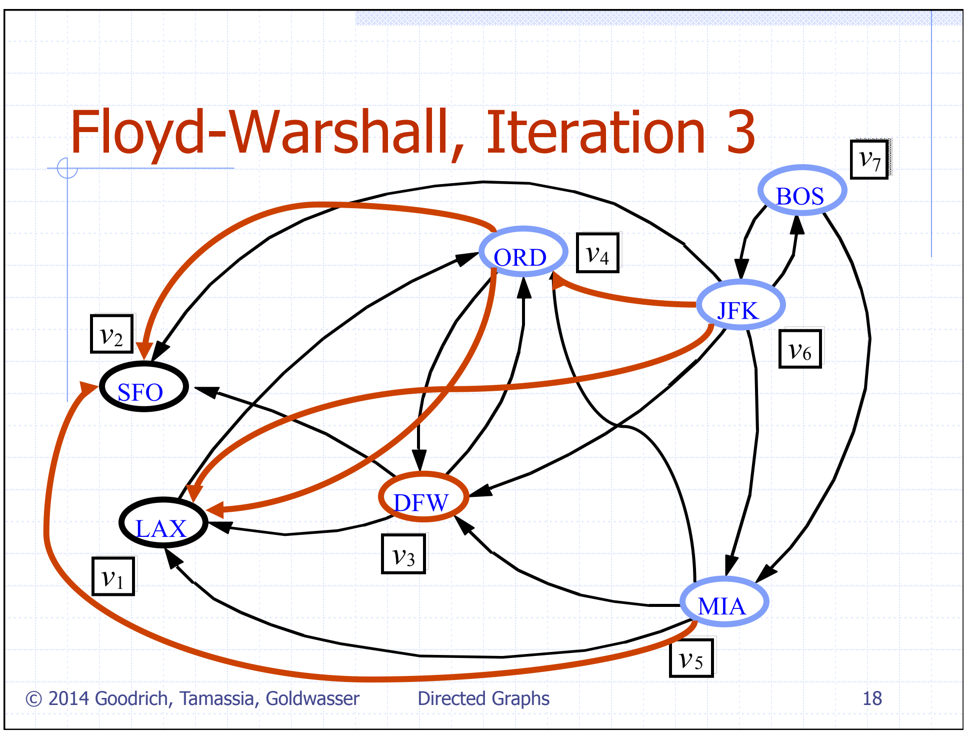
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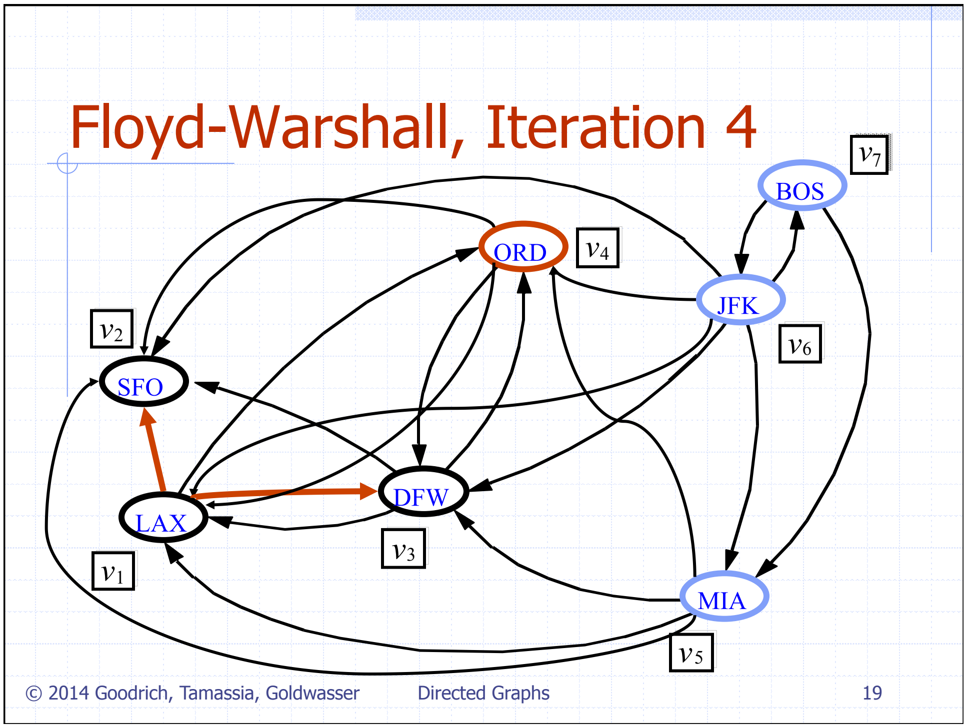
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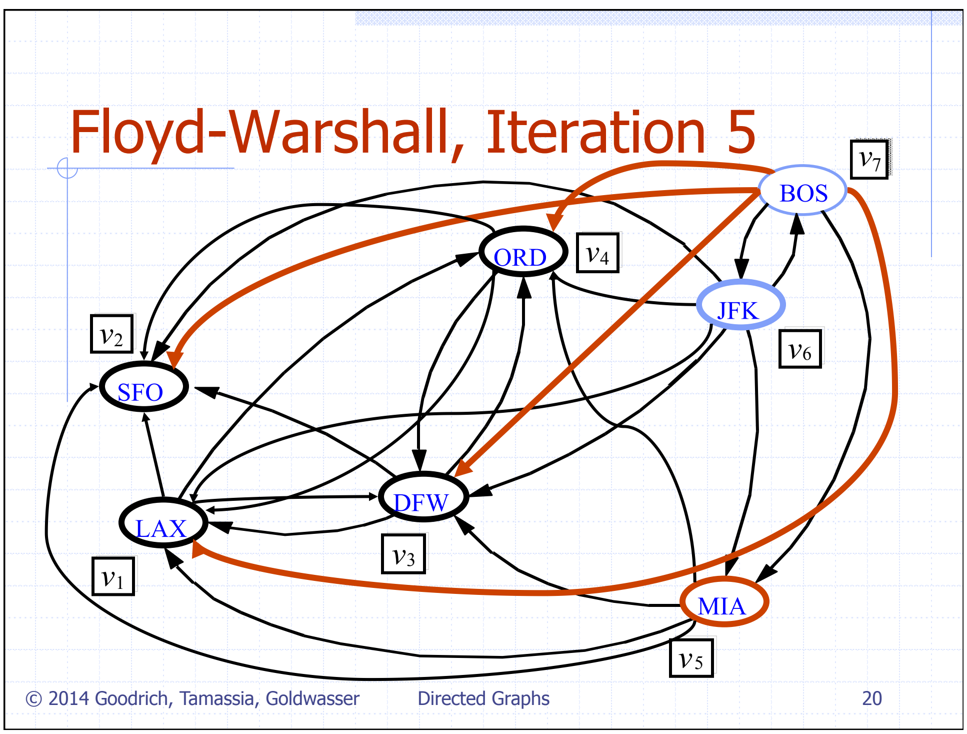
Floyd-Warshall, Iteration 5
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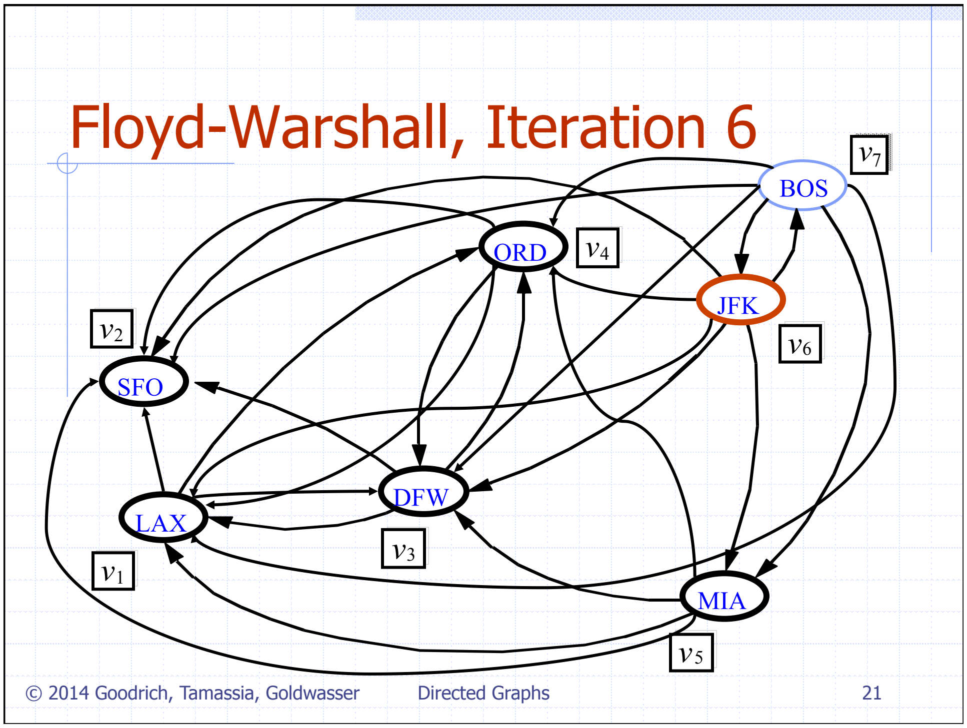
Floyd-Warshall, Iteration 6
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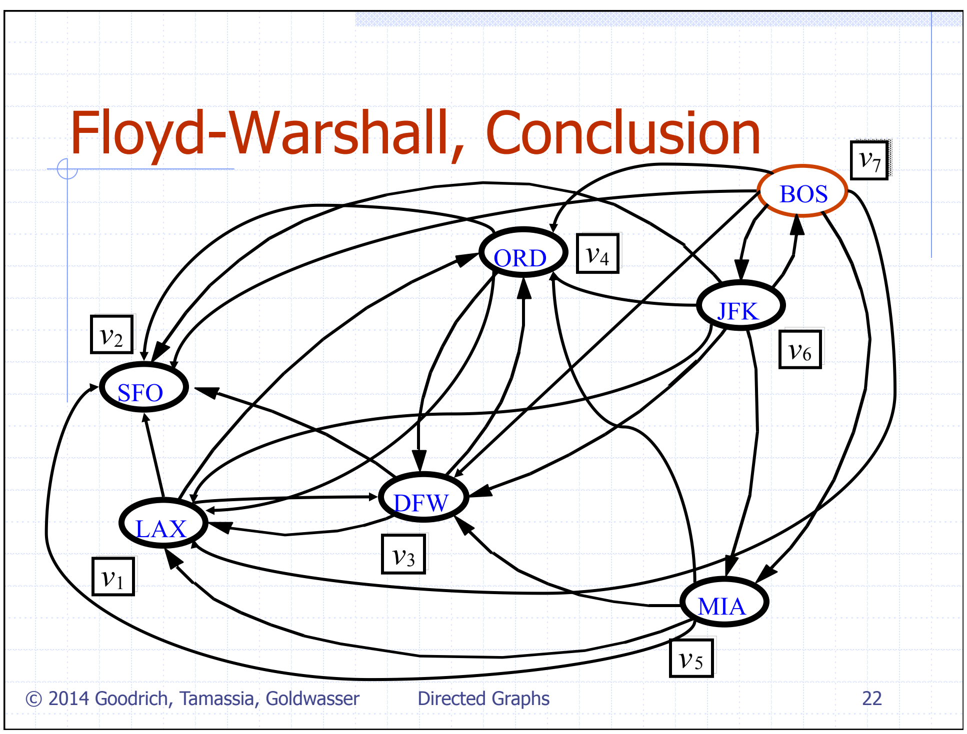
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LAX
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MIA
$v_5$
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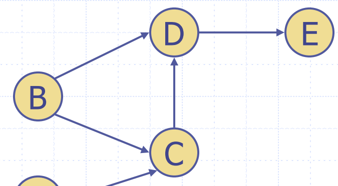
# DAGs and Topological Ordering

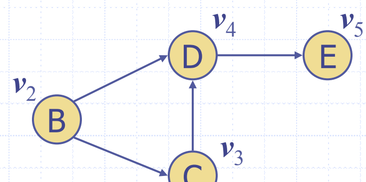
- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering
  $$v_1, \ldots, v_n$$
  of the vertices such that for every edge $(v_i, v_j)$, we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints

Theorem

A digraph admits a topological ordering if and only if it is a DAG



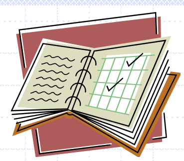

DAG $G$




Topological ordering of $G$

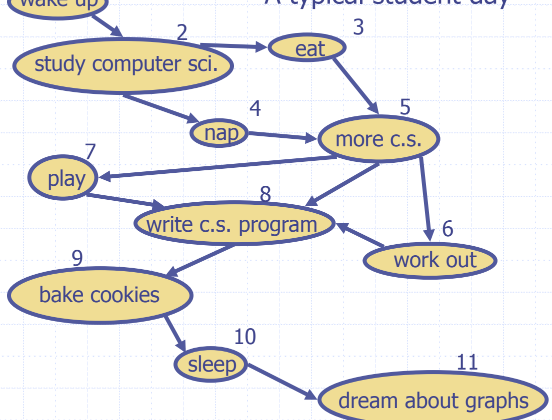
 

# Topological Sorting



- Number vertices, so that (u,v) in E implies u < v

A typical student day



# Algorithm for Topological Sorting

- Note: This algorithm is different than the one in the book

```
Algorithm TopologicalSort(G)
   H ← G          // Temporary copy of G
   n ← G.numVertices()
   while H is not empty do
        Let v be a vertex with no outgoing edges
        Label v ← n
        n ← n − 1
        Remove v from H
```

- Running time: O(n + m)



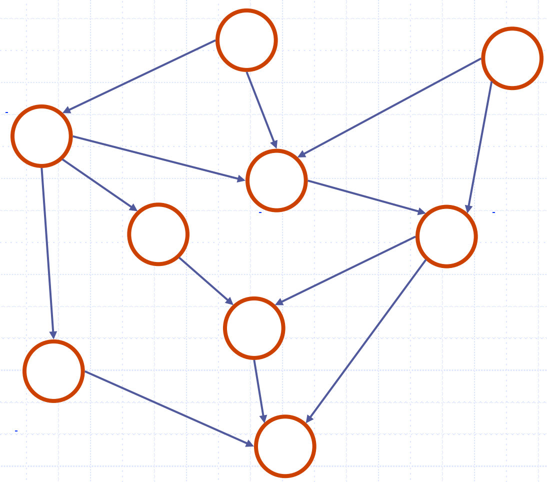
# Implementation with DFS

- Simulate the algorithm by using depth-first search
- O(n+m) time.

```
Algorithm topologicalDFS(G)
    Input dag G
    Output topological ordering of G
   n ← G.numVertices()
   for all u ∈ G.vertices()
      setLabel(u, UNEXPLORED)
   for all v ∈ G.vertices()
      if getLabel(v) = UNEXPLORED
         topologicalDFS(G, v)
```

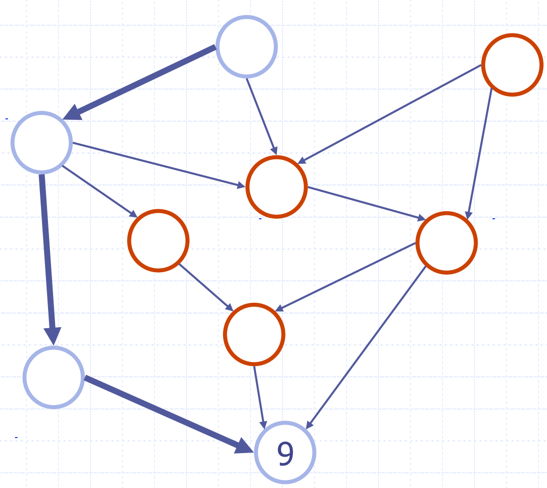
```
Algorithm topologicalDFS(G, v)
   Input graph G and a start vertex v of G
   Output labeling of the vertices of G
      in the connected component of v
   setLabel(v, VISITED)
   for all e ∈ G.outEdges(v)
      { outgoing edges }
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
         { e is a discovery edge }
         topologicalDFS(G, w)
      else
         { e is a forward or cross edge }
   Label v with topological number n
   n ← n - 1
```

## Topological Sorting Example





## Topological Sorting Example



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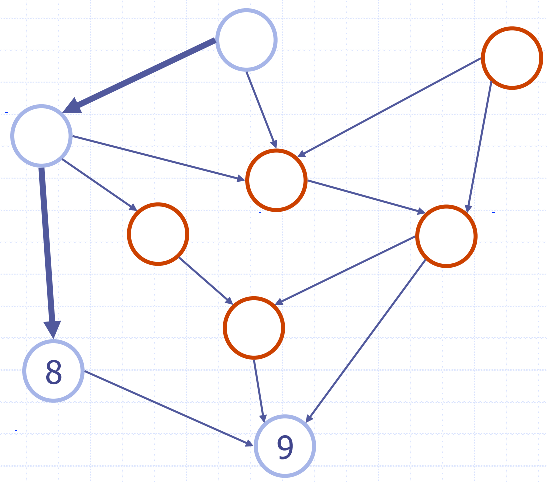




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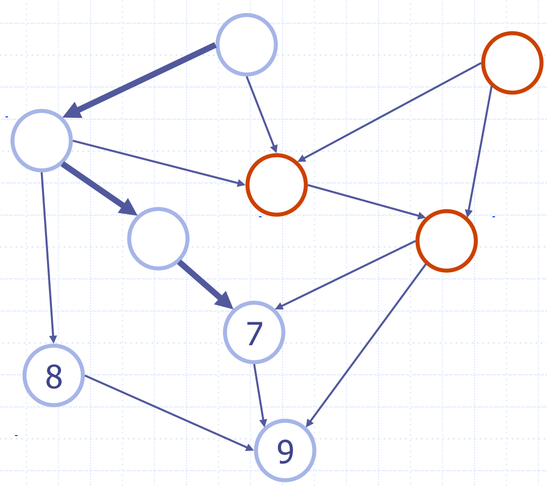

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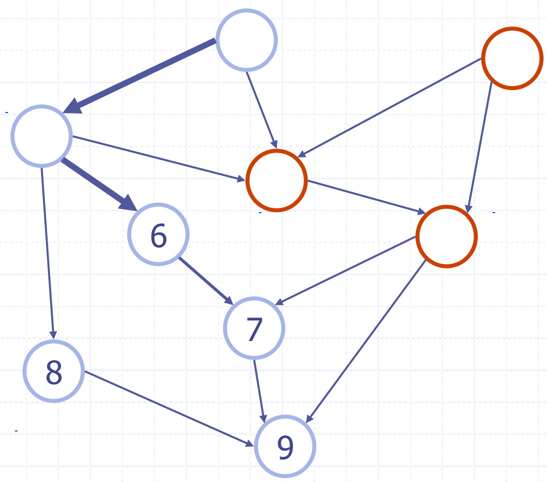




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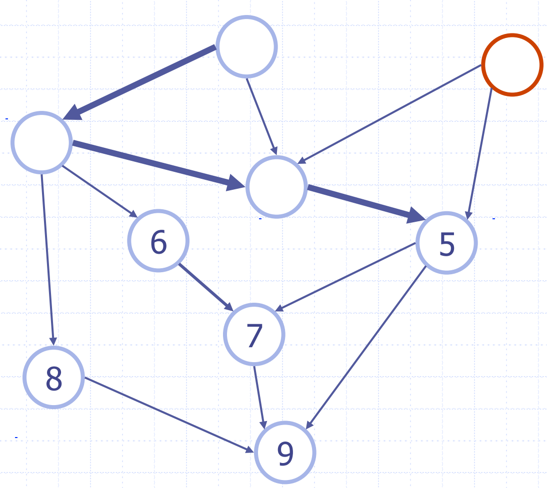
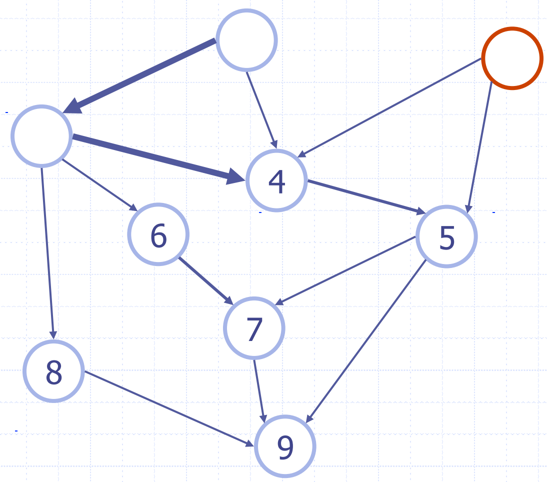
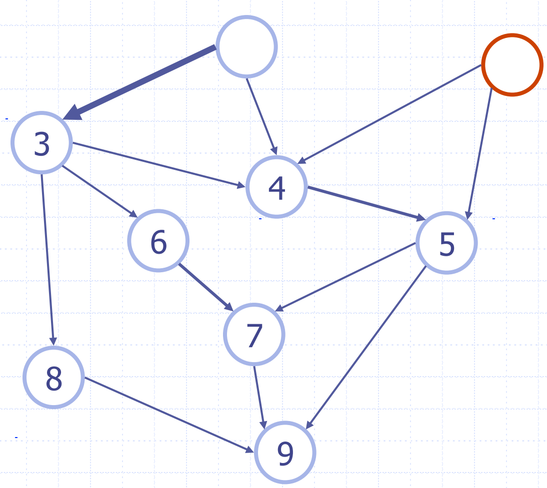


## Topological Sorting Example






## Topological Sorting Example



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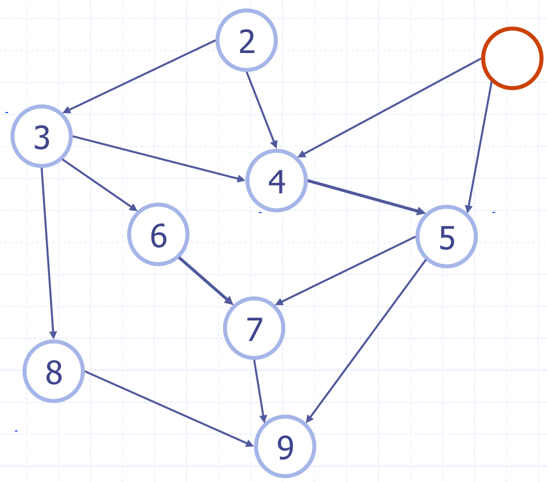


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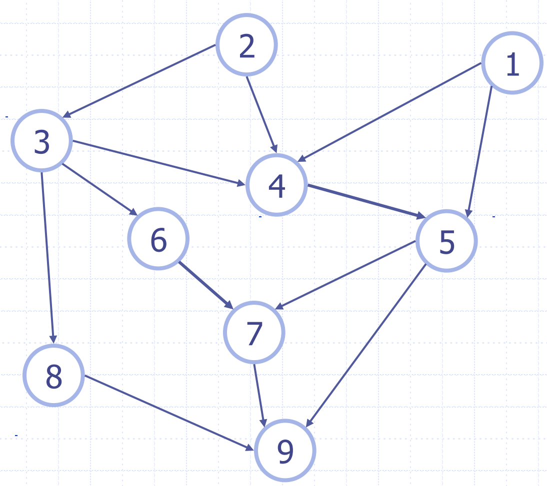
# Topological Sorting Example

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# Java Implementation

```
/** Returns a list of verticies of directed acyclic graph g in topological order. */
public static <V,E> PositionalList<Vertex<V>> topologicalSort(Graph<V,E> g) {
  // list of vertices placed in topological order
  PositionalList<Vertex<V>> topo = new LinkedPositionalList<>();
  // container of vertices that have no remaining constraints
  Stack<Vertex<V>> ready = new LinkedStack<>();
  // map keeping track of remaining in-degree for each vertex
  Map<Vertex<V>, Integer> inCount = new ProbeHashMap<>();
  for (Vertex<V> u : g.vertices()) {
    inCount.put(u, g.inDegree(u));               // initialize with actual in-degree
    if (inCount.get(u) == 0)                     // if u has no incoming edges,
      ready.push(u);                             // it is free of constraints
  }
  while (!ready.isEmpty()) {
    Vertex<V> u = ready.pop();
    topo.addLast(u);
    for (Edge<E> e : g.outgoingEdges(u)) {       // consider all outgoing neighbors of u
      Vertex<V> v = g.opposite(u, e);
      inCount.put(v, inCount.get(v) - 1);        // v has one less constraint without u
      if (inCount.get(v) == 0)
        ready.push(v);
    }
  }
  return topo;
}
```