Where the Theorems are Coming From? Lecture 1: The Case Study

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October 25, 2011.

Introduction and Schedule

Proposition

Mathematics is not about calculations. It is about definitions, theorems and proofs. Calculations are side-effect of mathematics.

Everybody, who is seriously interested in mathematics, has had a question: *where the theorems are coming from?*. Sometimes it is easier to prove a theorem, than formulate it. In these lectures I consider three methods for formulating mathematical hypotheses.

1. Lecture 1. Case study.
2. Lecture 2. Calculating the theorem,
3. Lecture 3. Counting mathematical structures.

The Consumer Survey

L.Ron Hubbard. Mission Earth, vol.3 p.87

...Oh, a survey. I haven‘t done a consumer survey.

He leaned forward and yelled through the mainly closed partition, "Bang-Bang! If you were a consumer, what would you really want to consume the most of?"

Bang-Bang skidded with screeching tires around a street-under-repair obstruction as he yelled back. "I‘ll let you in on something if you promise not to spread it around." He mounted a curb and got around a produce truck. "Everybody thinks I‘m called Bang-Bang because of explosives. That ain‘t so." He careened past a fire truck. Cherubino can tell you. I been called Bang-Bang since I was fourteen." He leaped the cab lightly over an open manhole cover. "The reason I‘m called Bang-Bang is because of girls. If Babe knew I was going in and out of the Gracious Palms, she‘d have a fit!"

"So the answer to the question of what you‘d consume the most of is girls."

"And girls and girls!" Bang-Bang yelled back, narrowly missing one on a crosswalk to prove his point.

Heller sat back. "Girls. Hm." He made a note on the inside back leaf of the marketing book, "*Survey done. Item: girls.*"

Watching Goats.

A group of scientists was walking on country-side and saw a herd of goats.

- Statistician: *There is a herd of white goats.*

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- Physicist: *There are eighteen white goats and one black.*
- Mathematician: *There is at least one goat with at least one black side.*
- Case study lady: *There are nineteen goats with one black and one white side.*

Boolean (Propositional) Formulae

Boolean variables are variables with a domain $\{0, 1\}$ (0=**false**, 1=**true**). We designate Boolean variables by x, y, z; with indexes, if convenient for our purposes.

Definition

1° Every Boolean variable and constants 0 või 1 are Boolean formulae.
2° IF A and B are Boolean formulae, then on $(\neg A)$, $(A \& B)$ and $(A \vee B)$ are Boolean formulae.

As usual, we fix the priorities of Boolean operations:

$$\neg(\text{highest priority}),\ \&,\ \vee$$

and allow to omit the parenthesis if they do not change the order of operations.

Disjunctive Normal Form (DNF)

Definition

Boolean formula F is in a *disjunctive normal form*, if it is a disjunction of conjunctions, i.e.:

$$F(x_1, \ldots, x_n) = \bigvee_{i=1}^{p} C_i,$$

where C_i is a *term*, which has the form:

$$C_i = \mathop{\&}_{j=1}^{m_i} l_{ij},$$

where l_{ij} is a *literal*. Literal is a variable or a negation of a variable.

Conjunctive Normal Form (CNF)

Definition

Boolean formula F is in a *conjunctive normal form*, if it is a conjunction of disjunctions, i.e.:

$$F(x_1, \ldots, x_n) = \mathop{\&}_{i=1}^{p} D_i$$

where D_i is a *disjunct*:

$$D_i = \bigvee_{j=1}^{m_i} l_{ij}$$

where l_{ij} is a *literal*.

Graph

Definition

Graph is a tuple $G = (V, E)$, where V is a finite set of vertices and $E \subseteq V \times V$ is a set of edges.

We will consider only *simple* graphs i.e. graphs without loops and multiple edges. A *complete graph* is a graph whose every two vertices are connected with an edge.

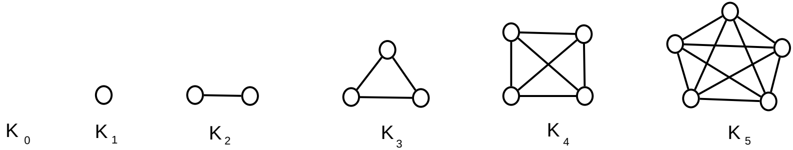


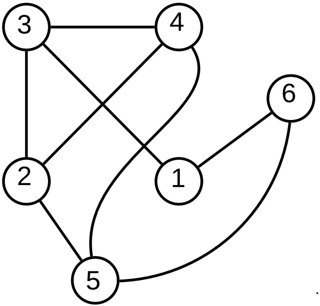
Figure: Complete graphs with 0,1,2,3,4 and 5 vertices.

We can suppose w.l.o.g. that $V = \{1, 2, \ldots, n\}$, where $n = |v|$.

Induced subgraph

Definition

Let $G = (V, E)$ be a graph and $V' \subseteq V$. *A subgraph of G, induced by the subset of vertices* V' is a graph $G' = (V', \{\{u, v\} : u \in V', v \in V', \{u, v\} \in E\})$.



Graph $G = (\{1, 2, 3, 4, 5, 6\}, E)$.

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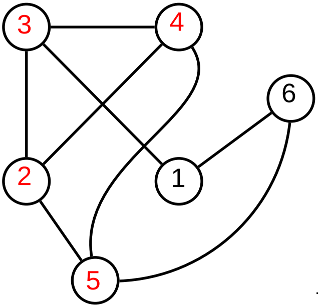

Graph $G = (\{1, 2, 3, 4, 5, 6\}, E)$. Let $V' = \{2, 3, 4, 5\}$.

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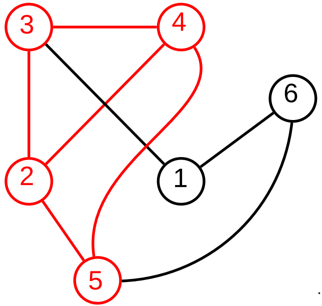

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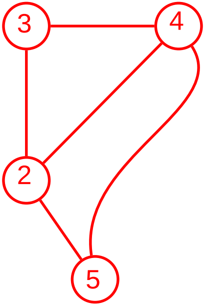


Graph $G = (\{1, 2, 3, 4, 5, 6\}, E)$. Let $V' = \{2, 3, 4, 5\}$.

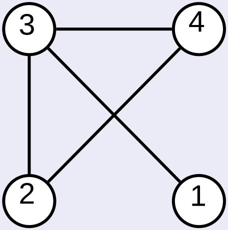
Clique

Definition

A *clique* of a graph $G = (V, E)$ is a subset of V, which induces a complete subgraph.

Example

Let G_1 be a graph:



All cliques of G_1 are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,3\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{2,3,4\}$. *Maximal cliques are* $\{2,3,4\}$ *and* $\{1,3\}$.

The Feynman Problem-Solving Algorithm

Algorithm

- *Write down the problem.*
- *Think very hard.*
- *Write down the answer.*

Write down the problem!

Problem

Find DNF and CNF which describe the structure of all cliques of a graph.

Every clique is a subset of $V = \{1, \ldots, n\}$ and can be characterized by its characteristic vector.

Definition

Let $V = \{1, \ldots, n\}$ and $V' \subseteq V$. A *characteristic vector* of V' is a binary vector $\chi_{V'} = \chi_1, \ldots, \chi_n$ such that

$$\chi_i = \begin{cases} 1, & \textit{if } i \in V', \\ 0, & \textit{if } i \notin V'. \end{cases}$$

We are looking for a DNF (CNF) $F_G(x_1, \ldots, x_n)$ such that $F_G(\chi_1, \ldots, \chi_n) = 1$ if and only if $\chi_1, \ldots, \chi_n$ is a characteristic vector of some clique of G. Let us start with a concrete case – graph G_1.

A truth-table for a function F_{G_1}, characterizing a clique structure of G_1.

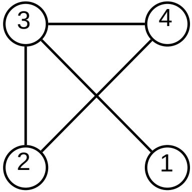


Figure: Graph G_1.

x_1	x_2	x_3	x_4	f_{G_1}	clique
0	0	0	0	1	$\emptyset$
0	0	0	1	1	$\{4\}$
0	0	1	0	1	$\{3\}$
0	0	1	1	1	$\{3,4\}$
0	1	0	0	1	$\{2\}$
0	1	0	1	1	$\{2,4\}$
0	1	1	0	1	$\{2,3\}$
0	1	1	1	1	$\{2,3,4\}$
1	0	0	0	1	$\{1\}$
1	0	0	1	0	–
1	0	1	0	1	$\{1,3\}$
1	0	1	1	0	–
1	1	0	0	0	–
1	1	0	1	0	–
1	1	1	0	0	–
1	1	1	1	0	–

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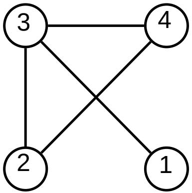

Figure: Graph G_1.

If we have a truth-table for a Boolean function F, we can easily write down a perfect DNF and perfect CNF for F.

x_1	x_2	x_3	x_4	f_{G_1}	clique
0	0	0	0	1	$\emptyset$
0	0	0	1	1	$\{4\}$
0	0	1	0	1	$\{3\}$
0	0	1	1	1	$\{3,4\}$
0	1	0	0	1	$\{2\}$
0	1	0	1	1	$\{2,4\}$
0	1	1	0	1	$\{2,3\}$
0	1	1	1	1	$\{2,3,4\}$
1	0	0	0	1	$\{1\}$
1	0	0	1	0	–
1	0	1	0	1	$\{1,3\}$
1	0	1	1	0	–
1	1	0	0	0	–
1	1	0	1	0	–
1	1	1	0	0	–
1	1	1	1	0	–

Set of terms for a perfect DNF of F_{G_1}.

x_1	x_2	x_3	x_4	f_{G_1}	term
0	0	0	0	1	$(\overline{x}_1 \& \overline{x}_2 \& \overline{x}_3 \& \overline{x}_4)$
0	0	0	1	1	$(\overline{x}_1 \& \overline{x}_2 \& \overline{x}_3 \& x_4)$
0	0	1	0	1	$(\overline{x}_1 \& \overline{x}_2 \& x_3 \& \overline{x}_4)$
0	0	1	1	1	$(\overline{x}_1 \& \overline{x}_2 \& x_3 \& x_4)$
0	1	0	0	1	$(\overline{x}_1 \& x_2 \& \overline{x}_3 \& \overline{x}_4)$
0	1	0	1	1	$(\overline{x}_1 \& x_2 \& \overline{x}_3 \& x_4)$
0	1	1	0	1	$(\overline{x}_1 \& x_2 \& x_3 \& \overline{x}_4)$
0	1	1	1	1	$(\overline{x}_1 \& x_2 \& x_3 \& x_4)$
1	0	0	0	1	$(x_1 \& \overline{x}_2 \& \overline{x}_3 \& \overline{x}_4)$
1	0	0	1	0	–
1	0	1	0	1	$(x_1 \& \overline{x}_2 \& x_3 \& \overline{x}_4)$
1	0	1	1	0	–
1	1	0	0	0	–
1	1	0	1	0	–
1	1	1	0	0	–
1	1	1	1	0	–

Perfect DNF for F_{G_1}.

$$(\overline{x}_1 \& \overline{x}_2 \& \overline{x}_3 \& \overline{x}_4) \vee$$
$$(\overline{x}_1 \& \overline{x}_2 \& \overline{x}_3 \& x_4) \vee$$
$$(\overline{x}_1 \& \overline{x}_2 \& x_3 \& \overline{x}_4) \vee$$
$$(\overline{x}_1 \& \overline{x}_2 \& x_3 \& x_4) \vee$$
$$(\overline{x}_1 \& x_2 \& \overline{x}_3 \& \overline{x}_4) \vee$$
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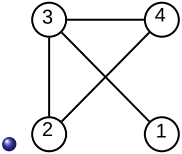
$$(x_1 \& \overline{x}_2 \& \overline{x}_3 \& \overline{x}_4) \vee$$
$$(x_1 \& \overline{x}_2 \& x_3 \& \overline{x}_4)$$

After minimizing we get

$$DF_{G_1} = (\overline{x}_2 \& \overline{x}_4) \vee (\overline{x}_1).$$

Think very hard!



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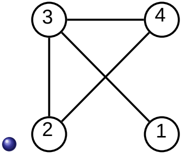
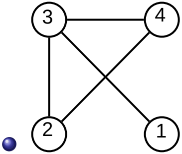


$$DF_{G_1} = (\overline{x}_2 \& \overline{x}_4) \vee (\overline{x}_1).$$

- We can see, that the set of variables in the first term of the DNF is the complement of the maximal clique $\{x_1, x_3\}$ and the set of variables in the second term is the complement of the maximal clique $\{x_2, x_3, x_4\}$. (remember, that there are exactly two maximal cliques in G_1).

Think very hard!



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- Let us make a courageous hypothesis, that it is not accidental. In general case every maximal clique $V' \subseteq V$ of G determines a term $T_{V'} = \&_{i \in V \setminus V'} \overline{x}_i$. Formula DF_G is a disjunction of all such terms:

$$DF_G = \bigvee_{V' = maxclique} T_{V'}.$$

Write down the answer!

Hypothesis

Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$. Binary vector $\chi \in \{0,1\}^n$ is the characteristic vector of the clique of G if and only if

$$DF_G(\chi) = \left[\bigvee_{V'=maxclique} \left(\underset{i \in V \setminus V'}{\&} \overline{x}_i \right) \right] (\chi) = 1.$$

For a hypothesis to became a theorem, it has to be proved.

The Theorem

Theorem

Let $G = (V, E)$ be a graph with vertex set $V = \{1, \dots, n\}$. A binary vector $\chi \in \{0, 1\}^n$ is a characteristic vector of the clique of G if and only if

$$DF_G(\chi) = \left[\bigvee_{V'=maxclique} \left(\underset{i \in V \setminus V'}{\&} \overline{x}_i \right) \right] (\chi) = 1.$$

Proof. 1. $\Longrightarrow$. Let $V' \subseteq V$ be a clique of a graph G and $\chi = (\chi_1, \dots, \chi_n)$ his characteristic vector. There exists a maximal cligue V'' such that $V' \subseteq V''$. Formula DF_G contains a term

$$T_{V''} = \underset{i \in V \setminus V''}{\&} \overline{x}_i.$$

Term $T_{V''}$ is obviously true for the characteristic vector β of V'', because $\beta_i = 0$ for every $i \in V \setminus V''$. V' is a subset of V'', therefore if $\beta_i = 0$, then $\chi_i = 0$. Consequently $T_{V''}(\chi) = 1$ and $DF_G(\chi) = 1$.

Second part of the proof.

2. $\Longleftarrow$. Let $\chi \in \{0,1\}^n$ be an assignment such that $DF_G(\chi) = 1$. Let $V_\chi \subseteq V$ be a subset of V whose characteristic vector is χ. We have to show, that V_χ is a clique of G. If $DF_G(\chi) = 1$, then there must be a term $T_{V''}(x)$ for some maximal clique of V'' of G, which is true for an assignment χ. It is possible only if $\chi_i = 0$ for every $i \in V \backslash V''$. It means, that $V_\chi \subseteq V''$ and, concequently, V_χ is a clique of G.

□

Let us do the same for calculating a CNF.

The set of disjuncts for a perfect CNF of F_{G_1}.

x_1	x_2	x_3	x_4	f_{G_1}	disjunct
0	0	0	0	1	–
0	0	0	1	1	–
0	0	1	0	1	–
0	0	1	1	1	–
0	1	0	0	1	–
0	1	0	1	1	–
0	1	1	0	1	–
0	1	1	1	1	–
1	0	0	0	1	–
1	0	0	1	0	$(\overline{x}_1 \vee x_2 \vee x_3 \vee \overline{x}_4)$
1	0	1	0	1	–
1	0	1	1	0	$(\overline{x}_1 \vee x_2 \vee \overline{x}_3 \vee \overline{x}_4)$
1	1	0	0	0	$(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee x_4)$
1	1	0	1	0	$(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_4)$
1	1	1	0	0	$(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_4)$
1	1	1	1	0	$(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_4)$

A perfect CNF for F_{G_1}

$$(\overline{x}_1 \vee x_2 \vee x_3 \vee \overline{x}_4)\,\&$$
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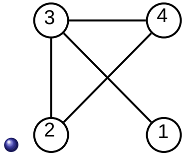
$$(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_4)\,\&$$
$$(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_4)\,\&$$
$$(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_4)$$

After minimizing we get the formula:

$$(\overline{x}_1 \vee \overline{x}_2)\,\&\,(\overline{x}_1 \vee \overline{x}_4).$$

Think once more!



$$(\overline{x}_1 \vee \overline{x}_2) \,\&\, (\overline{x}_1 \vee \overline{x}_4)$$

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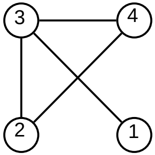

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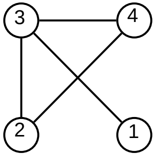
- We can see, that every disjunct corresponds to a missing edge of G_1.

Think once more!



$$(\overline{x}_1 \vee \overline{x}_2) \,\&\, (\overline{x}_1 \vee \overline{x}_4)$$

- We can see, that every disjunct corresponds to a missing edge of G_1.
- We can formulate a hypothesis – a general formula is:

$$CF_G = \underset{\{i,j\} \notin E}{\&} (\overline{x}_i \vee \overline{x}_j).$$

The Theorem.

Theorem

Let $G = (V, E)$ be a graph with vertex set $V = \{1, \dots, n\}$. A binary vector $\chi \in \{0,1\}^n$ is a characteristic vector of some clique of G if and only if

$$\left[\underset{\{i,j\} \notin E}{\&} (\overline{x}_i \vee \overline{x}_j) \right] (\chi) = 1.$$

Proof. 1. $\Longrightarrow$. Let $V' \subseteq V$ be a clique of a graph G and $\chi = (\chi_1, \dots, \chi_n)$ his characteristic vector. We have to show, that $CF_G(\chi) = 1$. Suppose to the contrary, that $CF_G(\chi) = 0$. Then at least one disjunct, let it be $\overline{x}_i \vee \overline{x}_j$, must have value 0 for an assignment χ. Then $\chi_i = 1$ and $\chi_j = 1$. If $\overline{x}_i \vee \overline{x}_j$ is a disjunct of CF_G, then $\{i,j\} \notin E$ and V' is not a clique of G. Contradiction.
2. $\Longleftarrow$. Suppose $CF_G(\chi) = 1$ for a characteristic vector $\chi = (\chi_1, \dots, \chi_n)$ of some $V' \subseteq V$. Suppose to the contrary, that V' is not a clique of G. Then there must exist vertices $i,j \in V'$ i.e. $\chi_i = 1, \chi_j = 1$ such that $\{i,j\} \notin E$. Then $\overline{x}_i \vee \overline{x}_j$ is a disjunct of CF_G which takes truth-value 0 for χ and $CF_G(\chi) = 0$. Contradiction. □

Some philosophical speculations.

Theorem

Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$. Binary vector $\chi \in \{0,1\}^n$ is the characteristic vector of the clique of G if and only if

$$DF_G(\chi) = \left[\bigvee_{V' = maxclique} \left(\underset{i \in V \setminus V'}{\&} \overline{x}_i \right) \right] (\chi) = 1.$$

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- The description of a problem using a CNF is similar to a law system of a democratic country – everything which is not forbidden is allowed.
- The *case study* is a common method for finding properties of discrete mathematical structures. Usually you have to consider much more than one case to formulate a hypothesis. For finding a bijection between certain classes of permutations and directed graphs I had to investigate 90 examples.
- Does these theorems have an application? A mathematician should not worry about applications. They give us nothing useful for a classical CLIQUE problem. The second theorem is just a polynomial reduction of NP-complete problem (CLIQUE) to another NP-complete problem (WEIGHTED-SAT).

- Nevertheless, about 20 years ago, when I formulated these theorems, using the *case study method* described above, I had some reasons. I had constructed a couple of algorithms for counting satisfying assignments of CNF. All the algorithms were implicitly generating a special (orthogonal) DNF for a given CNF. To prove, that my algorithms have an exponential worst case complexity I needed an example of CNF for which a minimal equivalent DNF has exponential size.

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- A CNF, constructed by Theorem 2 consists of $l \cdot k$ disjuncts and a DNF, constructed by Theorem 1 consists of k^l terms. Both formulae have only negated variables and by construction no term (disjunct) subsumes other. Therefore both CNF and DNF are minimal.

Mati Tombak. Keerukusteooria (Complexity theory). Tartu, 2007 (in Estonian).