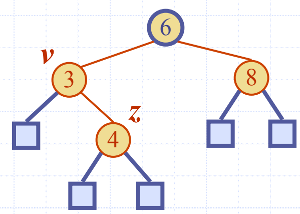
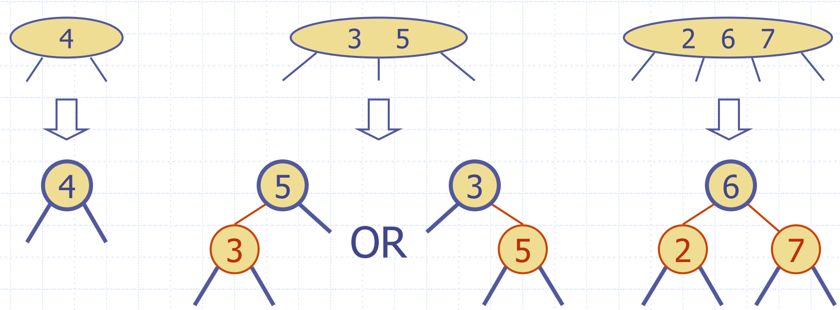
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

Red-Black Trees



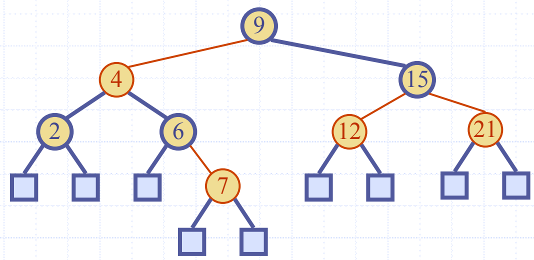
From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored red or **black**
- In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
  - simpler implementation with a single node type



Red-Black Trees

- A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - Root Property: the root is black
  - External Property: every leaf is black
  - Internal Property: the children of a red node are black
  - Depth Property: all the leaves have the same black depth



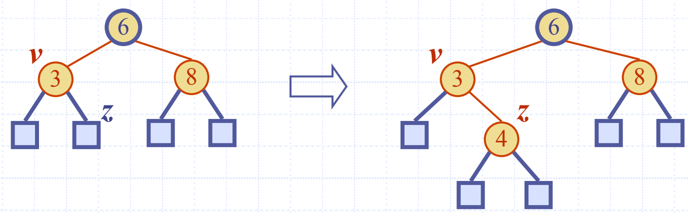
Height of a Red-Black Tree

- Theorem: A red-black tree storing n items has height $O(\log n)$

  Proof:
  - The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is $O(\log n)$
- The search algorithm for a binary search tree is the same as that for a binary search tree
- By the above theorem, searching in a red-black tree takes $O(\log n)$ time

Insertion

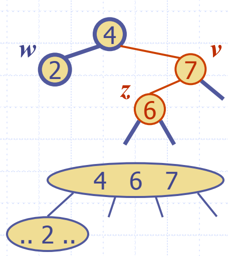
- To insert (k, o), we execute the insertion algorithm for binary search trees and color red the newly inserted node z unless it is the root
  - We preserve the root, external, and depth properties
  - If the parent v of z is black, we also preserve the internal property and we are done
  - Else (v is red) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree
- Example where the insertion of 4 causes a double red:



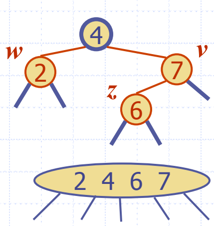
Remedying a Double Red

- Consider a double red with child z and parent v, and let w be the sibling of v

Case 1: w is black
- The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement

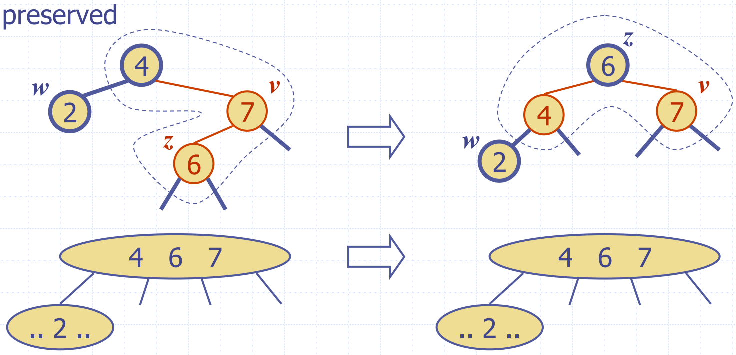


Case 2: w is red
- The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split



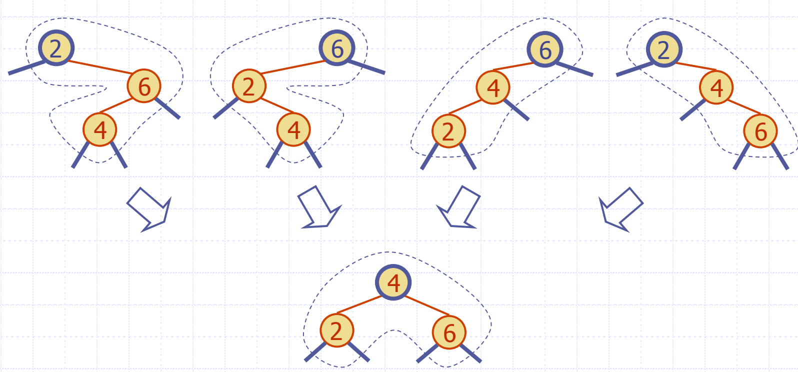
Restructuring

- A restructuring remedies a child-parent double red when the parent red node has a black sibling
- It is equivalent to restoring the correct replacement of a 4-node
- The internal property is restored and the other properties are preserved



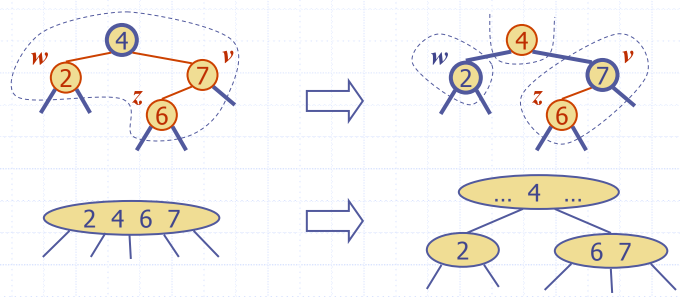
Restructuring (cont.)

- There are four restructuring configurations depending on whether the double red nodes are left or right children



Recoloring

- A recoloring remedies a child-parent double red when the parent red node has a red sibling
- The parent v and its sibling w become black and the grandparent u becomes red, unless it is the root
- It is equivalent to performing a split on a 5-node
- The double red violation may propagate to the grandparent u



Analysis of Insertion

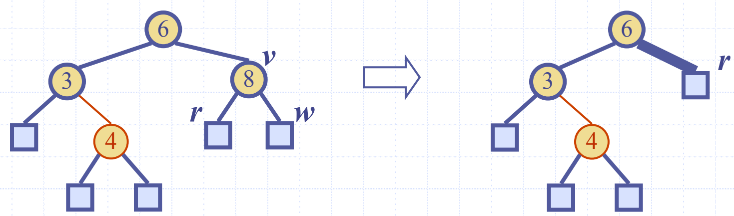
Algorithm *insert*(k, o)

1. We search for key k to locate the insertion node z
2. We add the new entry (k, o) at node z and color z red
3. **while** *doubleRed*(z)
   if *isBlack*(*sibling*(*parent*(z)))
   $z \leftarrow$ *restructure*(z)
   return
   else { *sibling*(*parent*(z)) is red }
   $z \leftarrow$ *recolor*(z)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most one restructuring taking $O(1)$ time
- Thus, an insertion in a red-black tree takes $O(\log n)$ time

Deletion

- To perform operation remove(k), we first execute the deletion algorithm for binary search trees
- Let v be the internal node removed, w the external node removed, and r the sibling of w
  - If either v of r was red, we color r black and we are done
  - Else (v and r were both black) we color r ***double black***, which is a violation of the internal property requiring a reorganization of the tree
- Example where the deletion of 8 causes a double black:



Remedying a Double Black

- The algorithm for remedying a double black node w with sibling y considers three cases

  Case 1: y is black and has a red child
  - We perform a restructuring, equivalent to a transfer , and we are done

  Case 2: y is black and its children are both black
  - We perform a recoloring, equivalent to a fusion, which may propagate up the double black violation

  Case 3: y is red
  - We perform an adjustment, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies
- Deletion in a red-black tree takes $O(\log n)$ time

Red-Black Tree Reorganization

Insertion	remedy double red	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	change of 4-node representation	double red removed
recoloring	split	double red removed or propagated up

Deletion	remedy double black	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	transfer	double black removed
recoloring	fusion	double black removed or propagated up
adjustment	change of 3-node representation	restructuring or recoloring follows

Java Implementation

```
/** An implementation of a sorted map using a red-black tree. */
public class RBTreeMap<K,V> extends TreeMap<K,V> {
  /** Constructs an empty map using the natural ordering of keys. */
  public RBTreeMap() { super(); }
  /** Constructs an empty map using the given comparator to order keys. */
  public RBTreeMap(Comparator<K> comp) { super(comp); }
  // we use the inherited aux field with convention that 0=black and 1=red
  // (note that new leaves will be black by default, as aux=0)
  private boolean isBlack(Position<Entry<K,V>> p) { return tree.getAux(p)==0;}
  private boolean isRed(Position<Entry<K,V>> p) { return tree.getAux(p)==1; }
  private void makeBlack(Position<Entry<K,V>> p) { tree.setAux(p, 0); }
  private void makeRed(Position<Entry<K,V>> p) { tree.setAux(p, 1); }
  private void setColor(Position<Entry<K,V>> p, boolean toRed) {
    tree.setAux(p, toRed ? 1 : 0);
  }
  /** Overrides the TreeMap rebalancing hook that is called after an insertion. */
  protected void rebalanceInsert(Position<Entry<K,V>> p) {
    if (!isRoot(p)) {
      makeRed(p);                  // the new internal node is initially colored red
      resolveRed(p);               // but this may cause a double-red problem
    }
  }
```

Java Implementation, 2

```
/** Remedies potential double-red violation above red position p. */
private void resolveRed(Position<Entry<K,V>> p) {
  Position<Entry<K,V>> parent,uncle,middle,grand; // used in case analysis
  parent = parent(p);
  if (isRed(parent)) {                          // double-red problem exists
    uncle = sibling(parent);
    if (isBlack(uncle)) {                       // Case 1: misshapen 4-node
      middle = restructure(p);                  // do trinode restructuring
      makeBlack(middle);
      makeRed(left(middle));
      makeRed(right(middle));
    } else {                                    // Case 2: overfull 5-node
      makeBlack(parent);                        // perform recoloring
      makeBlack(uncle);
      grand = parent(parent);
      if (!isRoot(grand)) {
        makeRed(grand);                         // grandparent becomes red
        resolveRed(grand);                      // recur at red grandparent
      }
    }
  }
}
```


Java Implementation, 3

```
/** Overrides the TreeMap rebalancing hook that is called after a deletion. */
protected void rebalanceDelete(Position<Entry<K,V>> p) {
  if (isRed(p))                                 // deleted parent was black
    makeBlack(p);                               // so this restores black depth
  else if (!isRoot(p)) {
    Position<Entry<K,V>> sib = sibling(p);
    if (isInternal(sib) && (isBlack(sib) || isInternal(left(sib))))
      remedyDoubleBlack(p);              // sib's subtree has nonzero black height
  }
}
```


Java Implementation, 4

```
/** Remedies a presumed double-black violation at the given (nonroot) position. */
private void remedyDoubleBlack(Position<Entry<K,V>> p) {
  Position<Entry<K,V>> z = parent(p);
  Position<Entry<K,V>> y = sibling(p);
  if (isBlack(y)) {
    if (isRed(left(y)) || isRed(right(y))) {          // Case 1: trinode restructuring
      Position<Entry<K,V>> x = (isRed(left(y)) ? left(y) : right(y));
      Position<Entry<K,V>> middle = restructure(x);
      setColor(middle, isRed(z)); // root of restructured subtree gets z's old color
      makeBlack(left(middle));
      makeBlack(right(middle));
    } else {                                           // Case 2: recoloring
      makeRed(y);
      if (isRed(z))
        makeBlack(z);                                  // problem is resolved
      else if (!isRoot(z))
        remedyDoubleBlack(z);                          // propagate the problem
    }
  } else {                                             // Case 3: reorient 3-node
    rotate(y);
    makeBlack(y);
    makeRed(z);
    remedyDoubleBlack(p);                              // restart the process at p
  }
}
```